

COMPLETELY POSITIVE MAPS ON MODULES, INSTRUMENTS, EXTREMALITY PROBLEMS, AND APPLICATIONS TO PHYSICS

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ABSTRACT. Convex sets of completely positive maps and positive semidefinite kernels are considered in the most general context of modules over C^* -algebras and a complete characterization of their extreme points is obtained. As a byproduct, we determine extreme quantum instruments, preparations, channels, and extreme autocorrelation functions. Various applications to quantum information and measurement theories are given. The structure of quantum instruments is analyzed thoroughly.

1. INTRODUCTION

There is no question about the importance of completely positive maps (instruments, POVMs, channels) in quantum information and measurement theories [4, 10, 18, 19, 28]. One could say that they form a core of modern mathematical analysis of quantum theory. For example, a normalized positive operator valued measure (POVM) describes the statistics of the outcomes of a quantum measurement and can be identified with a quantum observable.

In 1970, Davies and Lewis [11] introduced the concept of instrument which turned out to be crucial in developing quantum measurement theory since, besides measurement statistics, it also describes the state change due to a quantum measuring process. In 1984, Ozawa [31] proved that any completely positive (CP) instrument can be dilated to a quantum measuring process, that is, any instrument can be realized as a measurement model of a POVM. The role of complete positivity is essential in this characterization, see also [2]. Later Holevo [20] analyzed the structure of instruments and showed that any CP instrument has a pointwise Kraus decomposition [28]. Recently, applications of quantum instruments and their extremality problems have been studied extensively, see e.g. [5, 7, 9, 15, 17, 20, 37] and references therein.

Since instruments (and hence POVMs and channels) are special cases of CP maps, their mathematical analysis is based on the celebrated Stinespring dilation theorem [35]. This theorem has been generalized in many directions, the most general extension being [34, Theorem 4.3] where the CP maps are defined on a (unital) C^* -algebra and get their values in the vector space of A -sesquilinear A -valued forms on an A -module V (where A is a C^* -algebra). In quantum mechanics, one typically chooses $A = \mathbb{C}$ and let V be a Hilbert space, but in geometric theories of physics (general relativity, gauge field theory, etc.) one uses more complicated algebras A . For example, when theory is based on a vector bundle structure over a manifold Ω , one may take $A = C_0(\Omega)$, the continuous functions $\Omega \rightarrow \mathbb{C}$ vanishing at infinity. Then V could be the linear space of continuous vector fields. This type of structures (especially Hilbert C^* -modules [30]) are used in noncommutative geometry [13] which forms a link between geometric theories and quantum theory. Sometimes noncommutative geometry is viewed as a route to quantum gravity and spacetime. A nice application of CP maps to the problem of quantum spacetime is given in [16].

In this article, we define convex sets of CP maps and positive definite kernels in the most general context and characterize completely their extreme points. We apply this result e.g. to arbitrary CP instruments. Here are the results of this paper (some of them are known in

the discrete finite-dimensional cases but our results are also valid in ‘nondiscrete’ cases and in infinite dimensions):

- The structure of an arbitrary instrument \mathcal{M} is determined in several different ways (Theorem 2) by using e.g. structure vectors $\psi_m^t(x)$, generalized vectors $d_k^t(x)$, pointwise Kraus operators $A_k(x)$, and setwise Kraus operators $A_k(X)$. See also the Appendix.
- Any instrument has a *minimal* pointwise Kraus decomposition (item (2) of Theorem 2).
- The Dirac formalism is extended to instruments (Remark 3) so that it can be used to find compatible instruments of POVMs (Section 4).
- We characterize the extreme points of the convex set of instruments (see (4) and (5) of Theorem 2 and Remark 2).
- The extreme point characterization of instruments is applied to observables, preparations, the discrete case, channels, and the finite dimensional case (Subsection 3.3).
- Extreme instruments are discrete in finite dimensions (Proposition 2).
- For any POVM M , we show that the M -compatible instruments can be identified with the decomposable CP channels (Theorem 3) and can be viewed as combinations of Lüders operations and channels (Corollary 1).
- We present a complete characterization for pure realizations (measurement models) of instruments (Theorem 4 and Remark 6), and for minimal pure realizations (Corollary 2).
- The standard model of quantum measurement theory is generalized for arbitrary POVMs (Example 10).
- We determine the posterior (i.e. the post measurement) states (of a measurement) for arbitrary input states by using a minimal pointwise Kraus form of an instrument (Subsection 5.1).
- Any instrument can be maximally refined into a rank-1 instrument, and if an instrument is extreme then its rank-1 refinement is also extreme (Proposition 3).
- For any rank-1 POVM M , we prove that the M -compatible instruments are all nuclear and their associate channels are entanglement-breaking (Theorem 5 and Example 7).
- We determine the extreme points of the convex set of ‘very general’ positive definite kernels (Theorem 7) and, as an application, characterize extreme autocorrelation functions of stochastic processes (Proposition 4).
- A complete characterization of the extreme points of the convex set of ‘very general’ CP maps on modules is given in Theorem 9.
- Finally, we present a generalization for Choi isomorphism widely used in quantum information (Theorem 10).

2. BASIC NOTATIONS AND DEFINITIONS

For any Hilbert space \mathcal{H} we let $\mathcal{L}(\mathcal{H})$ [resp. $\mathcal{T}(\mathcal{H})$] denote the set of bounded [resp. trace-class] operators on \mathcal{H} . We let the innerproduct $\langle \cdot | \cdot \rangle$ of a Hilbert space (or any sesquilinear form) be linear with respect to its second argument. We say that a positive operator $\rho \in \mathcal{T}(\mathcal{H})$ of trace 1 is a state (or a density operator) and denote the set of states by $\mathcal{S}(\mathcal{H})$. The identity operator of any Hilbert space \mathcal{H} is denoted by $I_{\mathcal{H}}$. Throughout this article, we let \mathcal{H} and \mathcal{K} be *separable* (complex) nontrivial Hilbert spaces and (Ω, Σ) be a measurable space (i.e. Σ is a σ -algebra of subsets of a set Ω). If a fixed measure μ is given on (Ω, Σ) , without restricting generality, we assume that Σ is complete with respect to μ (i.e. contains μ -null sets). Hence, the concepts of Σ -measurability and μ -measurability coincide and we may just speak about measurability of a function $f : \Omega \rightarrow \mathbb{C}$. As usual we define an empty sum to be 0, e.g. $\sum_{k=1}^0 (\dots) := 0$, and $\mathbb{N} := \{0, 1, \dots\}$. Moreover, $\mathbb{N}_{\infty} := \mathbb{N} \cup \{\infty\}$ and $\mathbb{N}_{+} := \{1, 2, \dots\}$.

Operator measures. Let $\mathbf{M} : \Sigma \rightarrow \mathcal{L}(\mathcal{H})$ be an *operator (valued) measure*, i.e. (ultra)weakly σ -additive mapping. We call \mathbf{M} *positive* if for all $X \in \Sigma$, $\mathbf{M}(X) \geq 0$, *normalized* if $\mathbf{M}(\Omega) = I_{\mathcal{H}}$, and *projection valued* if $\mathbf{M}(X)^2 = \mathbf{M}(X)^* = \mathbf{M}(X)$ for all $X \in \Sigma$. Normalized positive operator valued measures (POVMs) are identified with (*quantum*) *observables* whereas normalized projection valued measures (PVMs) are called *spectral measures* or *sharp observables*. The convex set of POVMs $\mathbf{M} : \Sigma \rightarrow \mathcal{L}(\mathcal{H})$ is denoted by $\text{Obs}(\Sigma, \mathcal{H})$ and its extreme points by $\text{Ext Obs}(\Sigma, \mathcal{H})$. A convex combination (observable) $t\mathbf{M}_1 + (1-t)\mathbf{M}_2$, $0 < t < 1$, can be viewed as a randomization of measuring procedures represented by the observables \mathbf{M}_1 and \mathbf{M}_2 . An extreme observable $\mathbf{M} \in \text{Ext Obs}(\Sigma, \mathcal{H})$ cannot be obtained as a (nontrivial) combination; this means that the measurement of \mathbf{M} involves no redundancy caused by mixing different measuring schemes.

Instruments. We say that a map $\mathcal{M} : \Sigma \times \mathcal{L}(\mathcal{K}) \rightarrow \mathcal{L}(\mathcal{H})$ is a (*CP quantum*) *instrument* if

- (1) for all $X \in \Sigma$, the mapping $\mathcal{L}(\mathcal{K}) \ni B \mapsto \mathcal{M}(X, B) \in \mathcal{L}(\mathcal{H})$ is linear, *completely positive* (CP), and ultraweakly continuous (normal),
- (2) $\mathcal{M}(\Omega, I_{\mathcal{K}}) = I_{\mathcal{H}}$,
- (3) $\text{tr}[\rho \mathcal{M}(\cup_{i=1}^{\infty} X_i, B)] = \sum_{i=1}^{\infty} \text{tr}[\rho \mathcal{M}(X_i, B)]$ for any disjoint sequence $\{X_i\}_{i=1}^{\infty} \subseteq \Sigma$ and for all $\rho \in \mathcal{T}(\mathcal{H})$, $B \in \mathcal{L}(\mathcal{K})$.

For any $B \in \mathcal{L}(\mathcal{K})$, we define an operator measure

$$\mathbf{M}_B : \Sigma \rightarrow \mathcal{L}(\mathcal{H}), X \mapsto \mathbf{M}_B(X) := \mathcal{M}(X, B).$$

It is positive if $B \geq 0$ and normalized if $B = I_{\mathcal{K}}$. Hence, $\mathbf{M}_{I_{\mathcal{K}}}$ is a POVM, the *associate observable* of \mathcal{M} . An instrument can be seen as a certain collection of operator measures (indexed by bounded operators B). Any $B \in \mathcal{L}(\mathcal{K})$ can be (nonuniquely) decomposed into positive parts, that is, $B = \sum_{k=0}^3 i^k B_k$ where operators B_k are bounded and positive. Hence, by linearity, any instrument \mathcal{M} is uniquely determined already by positive operator measures \mathbf{M}_B where $B \geq 0$.

The convex set of instruments $\mathcal{M} : \Sigma \times \mathcal{L}(\mathcal{K}) \rightarrow \mathcal{L}(\mathcal{H})$ is denoted by $\text{Ins}(\Sigma, \mathcal{K}, \mathcal{H})$ and its extreme points by $\text{Ext Ins}(\Sigma, \mathcal{K}, \mathcal{H})$. Recall that any $\mathcal{M} \in \text{Ins}(\Sigma, \mathcal{K}, \mathcal{H})$ defines a ‘predual map’ $\mathcal{M}_* : \Sigma \times \mathcal{T}(\mathcal{H}) \rightarrow \mathcal{T}(\mathcal{K})$ by $\text{tr}[\mathcal{M}_*(X, \rho)B] := \text{tr}[\rho \mathcal{M}(X, B)]$ for all $X \in \Sigma$, $\rho \in \mathcal{T}(\mathcal{H})$, and $B \in \mathcal{L}(\mathcal{K})$. Sometimes \mathcal{M} is referred as a Heisenberg instrument and \mathcal{M}_* a Schrödinger instrument. Obviously, the map $\mathcal{M} \mapsto \mathcal{M}_*$ is an affine bijection so that \mathcal{M} is an extreme Heisenberg instrument if and only if \mathcal{M}_* is an extreme Schrödinger instrument.

Remark 1. On the first hand, any instrument \mathcal{M} defines a CP channel,¹ $B \mapsto \mathcal{M}(\Omega, B)$, the *associate channel* of \mathcal{M} . On the other hand, if $T : \mathcal{L}(\mathcal{K}) \rightarrow \mathcal{L}(\mathcal{H})$ is a CP channel, then by choosing $\Omega = \{0\}$ and $\Sigma = 2^{\{0\}} = \{\emptyset, \{0\}\}$, one can define an instrument $\mathcal{M}_T(\{0\}, B) := T(B)$, $B \in \mathcal{L}(\mathcal{K})$. Similarly, for any POVM $\mathbf{M} : \Sigma \rightarrow \mathcal{L}(\mathcal{H})$ there exist an instrument $\mathcal{M}^{\mathbf{M}} : \Sigma \times \mathbb{C} \rightarrow \mathcal{L}(\mathcal{H})$ defined by $\mathcal{M}^{\mathbf{M}}(X, c) := c\mathbf{M}(X)$ where $c \in \mathbb{C} \cong \mathcal{L}(\mathbb{C})$ (via $c \mapsto c|1\rangle\langle 1|$). We call the instruments \mathcal{M}_T and $\mathcal{M}^{\mathbf{M}}$ *trivial instruments* associated with T and \mathbf{M} , respectively. Thus, it follows that all general results for instruments are applicable to channels and POVMs.

Finally, recall the following nontrivial result of Davies and Lewis [11, Theorem 1]: For any POVM $\mathbf{M} \in \text{Obs}(\Sigma, \mathcal{H})$ there exists a (nonunique) CP instrument $\mathcal{M} \in \text{Ins}(\Sigma, \mathcal{H}, \mathcal{H})$, an *M-compatible instrument*, such that its associate observable $\mathbf{M}_{I_{\mathcal{H}}} = \mathbf{M}$, that is, $\text{tr}[\mathcal{M}_*(X, \rho)] \equiv \text{tr}[\rho \mathbf{M}(X)]$. Let $\mathcal{M} \in \text{Ins}(\Sigma, \mathcal{H}, \mathcal{H})$ be an *M-compatible instrument* of a projection valued measure $\mathbf{M} \in \text{Obs}(\Sigma, \mathcal{H})$. Then $\mathcal{M}(X, B) \equiv \mathbf{M}(X)T(B)$ where T is a CP channel such that $\mathbf{M}(X)T(B) \equiv T(B)\mathbf{M}(X)$ [31, Prop. 4.3 and 4.4]. It is then obvious that for a POVM \mathbf{M} and a

¹A map $T : \mathcal{L}(\mathcal{K}) \rightarrow \mathcal{L}(\mathcal{H})$ is a (*quantum*) *operation* if it is linear, positive, ultraweakly continuous, and $T(I_{\mathcal{K}}) \leq I_{\mathcal{H}}$. If, moreover, $T(I_{\mathcal{K}}) = I_{\mathcal{H}}$ then T is said to be a (*quantum*) *channel*. The maps $B \mapsto \mathcal{M}(X, B)$ are CP operations.

channel T there does not necessarily exist an instrument \mathcal{M} such that $\mathcal{M}(X, I_{\mathcal{K}}) \equiv \mathbf{M}(X)$ and $\mathcal{M}(\Omega, B) \equiv T(B)$.

3. DIAGONALIZATION AND EXTREMALITY RESULTS FOR POVMs AND INSTRUMENTS

Let $\mathbf{h} = \{h_n\}_{n=1}^{\dim \mathcal{H}}$ be an orthonormal (ON) basis of \mathcal{H} and

$$V_{\mathbf{h}} := \text{lin}_{\mathbb{C}}\{h_n \mid 1 \leq n < \dim \mathcal{H} + 1\}.$$

Note that $V_{\mathbf{h}}$ is dense in \mathcal{H} . Let $V_{\mathbf{h}}^{\times}$ be the algebraic antidual of the vector space $V_{\mathbf{h}}$. Recall that $V_{\mathbf{h}}^{\times}$ can be identified with the linear space of formal series $c = \sum_{n=1}^{\dim \mathcal{H}} c_n h_n$ where c_n 's are arbitrary complex numbers. Hence, $V_{\mathbf{h}} \subseteq \mathcal{H} \subseteq V_{\mathbf{h}}^{\times}$. Denote the dual pairing $\langle \psi | c \rangle := \sum_{n=1}^{\dim \mathcal{H}} \langle \psi | h_n \rangle c_n$ and $\langle c | \psi \rangle := \overline{\langle \psi | c \rangle}$ for all $\psi \in V_{\mathbf{h}}$ and $c \in V_{\mathbf{h}}^{\times}$. We say that a mapping $c : \Omega \rightarrow V_{\mathbf{h}}^{\times}$, $x \mapsto \sum_{n=1}^{\dim \mathcal{H}} c_n(x) h_n$ is *(weak*-)-measurable* if its components $x \mapsto c_n(x)$ are measurable. Note that, if $c : \Omega \rightarrow \mathcal{H} \subseteq V_{\mathbf{h}}^{\times}$ is weak*-measurable then the maps $x \mapsto \langle \psi | c(x) \rangle$ are measurable for all $\psi \in \mathcal{H}$. For any linear map $A : V_{\mathbf{h}} \rightarrow H$, where H is a Hilbert space, we let A^* denote the adjoint (transpose) linear map from H to $V_{\mathbf{h}}^{\times}$ defined by $\langle \psi | A^* \varphi \rangle := \langle A \psi | \varphi \rangle$, $\psi \in V_{\mathbf{h}}$, $\varphi \in H$. Note that A^* is not necessarily the usual Hilbert space adjoint of A given by the Fréchet-Riesz representation theorem. However, for bounded operators between Hilbert spaces and for elements of C^* -algebras we use the same symbol $*$ for the usual adjoint and involution.

Let \mathcal{H}_{\oplus} denote a direct integral $\int_{\Omega}^{\oplus} \mathcal{H}_{n(x)} d\mu(x)$ of separable Hilbert spaces $\mathcal{H}_{n(x)}$ such that $\dim \mathcal{H}_{n(x)} = n(x) \in \mathbb{N}_{\infty}$; here μ is a σ -finite nonnegative measure² on (Ω, Σ) . Let \mathcal{L} be an infinite dimensional separable Hilbert space with an ON basis $\mathbf{b} = \{b_n\}_{n=1}^{\infty}$. By choosing a measurable field of ON bases, $\{b_n(x)\}_{n=1}^{n(x)}$, one gets a decomposable unitary operator $U : \mathcal{H}_{\oplus} \rightarrow \mathcal{H}'_{\oplus}$, $U(x)b_n(x) := b_n$, where $\mathcal{H}'_{\oplus} := \int_{\Omega}^{\oplus} \mathcal{H}'_{n(x)} d\mu(x)$ with fibers

$$\mathcal{H}'_{n(x)} := \text{lin}_{\mathbb{C}}\{b_n \mid 1 \leq n \leq n(x)\}$$

if $0 < n(x) < \infty$, $\mathcal{H}'_0 := \{0\}$, and $\mathcal{H}'_{\infty} := \mathcal{L}$. Hence, without restricting generality, we simply assume that $\mathcal{H}_{n(x)} \equiv \mathcal{H}'_{n(x)}$ and thus $\mathcal{H}_{\oplus} = \mathcal{H}'_{\oplus}$. Now \mathcal{H}_{\oplus} can be considered as a closed subspace of $L^2(\mu, \mathcal{L}) \cong L^2(\mu) \otimes \mathcal{L}$, the μ -square integrable functions $\Omega \rightarrow \mathcal{L}$, and one has a decomposable projection $P = \int_{\Omega}^{\oplus} P(x) d\mu(x)$ from $L^2(\mu, \mathcal{L})$ onto \mathcal{H}_{\oplus} , where $P(x) = \sum_{n=1}^{n(x)} |b_n\rangle \langle b_n|$. We say that \mathcal{H}_{\oplus} is embedded in $L^2(\mu, \mathcal{L})$ and write $\mathcal{H}_{\oplus} \subseteq L^2(\mu, \mathcal{L})$.

For each $f \in L^{\infty}(\mu)$, we denote briefly by \hat{f} the multiplicative (i.e. diagonalizable) bounded operator $(\hat{f}\psi)(x) := f(x)\psi(x)$ on any direct integral Hilbert space $\mathcal{H}_{\oplus} = \int_{\Omega}^{\oplus} \mathcal{H}_{n(x)} d\mu(x)$. Especially, one has the *canonical spectral measure* $\Sigma \ni X \mapsto \hat{\chi}_X \in \mathcal{L}(\mathcal{H}_{\oplus})$ (where χ_X is the characteristic function of $X \in \Sigma$). We will use the following proposition [12, Theorem 1, p. 187] several times:

Proposition 1. *Let $\mathcal{H}_{\oplus}^i = \int_{\Omega}^{\oplus} \mathcal{H}_{n^i(x)}^i d\mu(x)$, $i = 1, 2$, be two direct integral Hilbert spaces, and let $D : \mathcal{H}_{\oplus}^1 \rightarrow \mathcal{H}_{\oplus}^2$ be a bounded operator. Then, $D\hat{\chi}_X = \hat{\chi}_X D$ for all $X \in \Sigma$ if and only if $D\hat{f} = \hat{f}D$ for all $f \in L^{\infty}(\mu)$ if and only if D is decomposable, i.e. $D = \int_{\Omega}^{\oplus} D(x) d\mu(x)$ where $D(x) : \mathcal{H}_{n^1(x)}^1 \rightarrow \mathcal{H}_{n^2(x)}^2$ are bounded and, for any $\psi \in \mathcal{H}_{\oplus}^1$, $(D\psi)(x) = D(x)\psi(x)$ for μ -almost all $x \in \Omega$, and $\|D\| = \mu\text{-ess sup}_{x \in \Omega} \|D(x)\| < \infty$.*

²Note that μ can be a probability measure everywhere in this paper; any σ -finite measure is equivalent with a probability measure.

3.1. Observables. We have the following theorem proved in [25, 33]:

Let $\mathbf{M} : \Sigma \rightarrow \mathcal{L}(\mathcal{H})$ be a positive operator measure and $\mu : \Sigma \rightarrow [0, \infty]$ a σ -finite measure such that \mathbf{M} is absolutely continuous with respect to μ . Let \mathbf{h} be an ON basis of \mathcal{H} .

Theorem 1. *There exists a direct integral $\mathcal{H}_\oplus = \int_\Omega^\oplus \mathcal{H}_{n(x)} d\mu(x)$ (with $n(x) \leq \dim \mathcal{H}$) such that, for all $X, X' \in \Sigma$,*

- (1) $\mathbf{M}(X) = Y^* \hat{\chi}_X Y$ where $Y = \sum_{m=1}^{\dim \mathcal{H}} |\psi_m\rangle \langle h_m|$ is a bounded operator and $\{\psi_m\}_{m=1}^{\dim \mathcal{H}} \subseteq \mathcal{H}_\oplus$ is such that the set of linear combinations of vectors $\chi_X \psi_m$ is dense in \mathcal{H}_\oplus (a minimal Naimark dilation for \mathbf{M}). Hence, by defining $Y(X) := \hat{\chi}_X Y = \sum_{m=1}^{\dim \mathcal{H}} |\chi_X \psi_m\rangle \langle h_m|$,

$$\mathbf{M}(X \cap X') = Y(X)^* Y(X') = \sum_{n,m=1}^{\dim \mathcal{H}} \int_{X \cap X'} \langle \psi_n(x) | \psi_m(x) \rangle d\mu(x) |h_n\rangle \langle h_m|$$

weakly (a minimal Kolmogorov decomposition for \mathbf{M}).

- (2) There are measurable maps $d_k : \Omega \rightarrow V_{\mathbf{h}}^\times$ such that, for all $x \in \Omega$, the vectors $d_k(x) \neq 0$, $k < n(x) + 1$ are linearly independent, and

$$\langle \varphi | \mathbf{M}(X) \psi \rangle = \int_X \sum_{k=1}^{n(x)} \langle \varphi | d_k(x) \rangle \langle d_k(x) | \psi \rangle d\mu(x), \quad \varphi, \psi \in V_{\mathbf{h}},$$

(a minimal diagonalization of \mathbf{M}).

- (3) \mathbf{M} is normalized if and only if $\{\psi_m\}_{m=1}^{\dim \mathcal{H}}$ is an ON set of \mathcal{H}_\oplus . Then Y is an isometry.
 (4) \mathbf{M} is a spectral measure if and only if $\{\psi_m\}_{m=1}^{\dim \mathcal{H}}$ is an ON basis of \mathcal{H}_\oplus . Then Y is a unitary operator and \mathcal{H}_\oplus can be identified with \mathcal{H} .
 (5) Let $\mathbf{M} \in \text{Obs}(\Sigma, \mathcal{H})$. Then $\mathbf{M} \in \text{Ext Obs}(\Sigma, \mathcal{H})$ if and only if, for any decomposable operator $D = \int_\Omega^\oplus D(x) d\mu(x) \in \mathcal{L}(\mathcal{H}_\oplus)$, the condition $Y^* D Y = 0$ implies $D = 0$.

By using the embedding $\mathcal{H}_\oplus \subseteq L^2(\mu, \mathcal{L})$ and an ON basis \mathbf{b} of \mathcal{L} , the relation between vectors $\psi_n(x)$ and $d_k(x)$ can be chosen to be $\langle \psi_n(x) | b_k \rangle = \langle h_n | d_k(x) \rangle$ so that we may then write

$$\langle \varphi | \mathbf{M}(X) \psi \rangle = \int_X \langle \varphi | A^1(x)^* A^1(x) \psi \rangle d\mu(x) = \int_X \sum_{k=1}^{n(x)} \langle \varphi | A_k(x)^* A_k(x) \psi \rangle d\mu(x), \quad \varphi, \psi \in V_{\mathbf{h}},$$

where $A^1(x) := \sum_{k=1}^{n(x)} |b_k\rangle \langle d_k(x)| = \sum_{n=1}^{\dim \mathcal{H}} |\psi_n(x)\rangle \langle h_n|$ and $A_k(x) := |1\rangle \langle d_k(x)|$ are (possibly unbounded) operators $V_{\mathbf{h}} \rightarrow \mathcal{H}_{n(x)}$ and $V_{\mathbf{h}} \rightarrow \mathbb{C}$, respectively. In addition,

$$A^1(x)^* A^1(x) = \sum_{k=1}^{n(x)} A_k(x)^* A_k(x) = \sum_{k=1}^{n(x)} |d_k(x)\rangle \langle d_k(x)| = \sum_{n,m=1}^{\dim \mathcal{H}} \langle \psi_n(x) | \psi_m(x) \rangle |h_n\rangle \langle h_m|$$

is an operator $V_{\mathbf{h}} \rightarrow V_{\mathbf{h}}^\times$ (or a sesquilinear form $V_{\mathbf{h}} \times V_{\mathbf{h}} \rightarrow \mathbb{C}$). Also one sees that $\mathbf{M} \in \text{Ext Obs}(\Sigma, \mathcal{H})$ if and only if, for any decomposable operator $D = \int_\Omega^\oplus D(x) d\mu(x) \in \mathcal{L}(\mathcal{H}_\oplus)$, the condition

$$\int_\Omega \langle \varphi | A^1(x)^* D(x) A^1(x) \psi \rangle d\mu(x) = 0, \quad \varphi, \psi \in V_{\mathbf{h}},$$

implies $D = 0$.

Let $\mathcal{M}^{\mathbf{M}} : \Sigma \times \mathbb{C} \rightarrow \mathcal{L}(\mathcal{H})$ be the trivial instrument associated with a POVM \mathbf{M} . Then, for example,

$$\langle \varphi | \mathcal{M}^{\mathbf{M}}(X, c) \psi \rangle = \int_X \sum_{k=1}^{n(x)} \langle \varphi | A_k(x)^* c A_k(x) \psi \rangle d\mu(x), \quad \varphi, \psi \in V_{\mathbf{h}}, \quad X \in \Sigma, \quad c \in \mathbb{C},$$

so that we have obtained a minimal pointwise Kraus form for \mathcal{M}^M . The next theorem generalizes the above constructions to arbitrary instruments.

3.2. Instruments. Let $\mathcal{M} : \Sigma \times \mathcal{L}(\mathcal{K}) \rightarrow \mathcal{L}(\mathcal{H})$ be an instrument and $\mu : \Sigma \rightarrow [0, \infty]$ a σ -finite measure such that $M_{I_{\mathcal{K}}}$ is absolutely continuous with respect to μ .³ Let $\mathbf{h} = \{h_n\}$ [resp. $\mathbf{k} = \{k_s\}$] be an ON basis of \mathcal{H} [resp. \mathcal{K}]. Denote $B_{st} := \langle k_s | B k_t \rangle$ for all $B \in \mathcal{L}(\mathcal{K})$ and $1 \leq s, t < \dim \mathcal{K} + 1$.

Theorem 2. *There exists a direct integral $\mathcal{H}_{\oplus} = \int_{\Omega}^{\oplus} \mathcal{H}_{n(x)} d\mu(x)$ (with $n(x) \leq \dim \mathcal{H} \dim \mathcal{K}$) such that, for all $X, X' \in \Sigma$ and $B, B' \in \mathcal{L}(\mathcal{K})$,*

$$(1) \mathcal{M}(X, B) = Y^*(B \otimes \hat{\chi}_X)Y \text{ where } Y : \mathcal{H} \rightarrow \mathcal{K} \otimes \mathcal{H}_{\oplus},$$

$$Y = \sum_{m=1}^{\dim \mathcal{H}} \sum_{t=1}^{\dim \mathcal{K}} |k_t \otimes \psi_m^t\rangle \langle h_m|,$$

is an isometry (i.e. $\sum_{t=1}^{\dim \mathcal{K}} \langle \psi_n^t | \psi_m^t \rangle = \delta_{nm}$) and $\{\psi_m^t\}_{m,t} \subseteq \mathcal{H}_{\oplus}$ is such that the set of linear combinations of vectors $\chi_X \psi_m^t$, $X \in \Sigma$, $m < \dim \mathcal{H} + 1$, $t < \dim \mathcal{K} + 1$, is dense in \mathcal{H}_{\oplus} (a minimal Stinespring dilation for \mathcal{M}). Hence, by defining $Y(X, B) := (B \otimes \hat{\chi}_X)Y = \sum_{m=1}^{\dim \mathcal{H}} \sum_{t=1}^{\dim \mathcal{K}} |(B k_t) \otimes (\chi_X \psi_m^t)\rangle \langle h_m|$,

$$\begin{aligned} \mathcal{M}(X \cap X', B^* B') &= Y(X, B)^* Y(X', B') \\ &= \sum_{n,m=1}^{\dim \mathcal{H}} \sum_{s,t=1}^{\dim \mathcal{K}} (B^* B')_{st} \int_{X \cap X'} \langle \psi_n^s(x) | \psi_m^t(x) \rangle d\mu(x) |h_n\rangle \langle h_m| \end{aligned}$$

weakly (a minimal Kolmogorov decomposition for \mathcal{M}).

(2) *The are measurable maps $d_k^t : \Omega \rightarrow V_{\mathbf{h}}^{\times}$ such that, for all $x \in \Omega$, the operators*

$$A_k(x) := \sum_{t=1}^{\dim \mathcal{K}} |k_t\rangle \langle d_k^t(x)|, \quad 1 \leq k < n(x) + 1,$$

from $V_{\mathbf{h}}$ to \mathcal{K} are linearly independent, and

$$\begin{aligned} \langle \varphi | \mathcal{M}(X, B) \psi \rangle &= \sum_{s,t=1}^{\dim \mathcal{K}} B_{st} \int_X \sum_{k=1}^{n(x)} \langle \varphi | d_k^s(x) \rangle \langle d_k^t(x) | \psi \rangle d\mu(x) \\ &= \int_X \sum_{k=1}^{n(x)} \langle \varphi | A_k(x)^* B A_k(x) \psi \rangle d\mu(x), \quad \varphi, \psi \in V_{\mathbf{h}}, \end{aligned}$$

(a minimal pointwise Kraus form of \mathcal{M}). By defining operators

$$A^t(x) := \sum_{k=1}^{n(x)} |b_k\rangle \langle d_k^t(x)|, \quad 1 \leq t < \dim \mathcal{K} + 1, \quad x \in \Omega,$$

from $V_{\mathbf{h}}$ to $\mathcal{H}_{n(x)}$ one gets

$$\langle \varphi | \mathcal{M}(X, B) \psi \rangle = \sum_{s,t=1}^{\dim \mathcal{K}} B_{st} \int_X \langle \varphi | A^s(x)^* A^t(x) \psi \rangle d\mu(x), \quad \varphi, \psi \in V_{\mathbf{h}}.$$

³ If the POVM $M_{I_{\mathcal{K}}}$ associated to an instrument \mathcal{M} is absolutely continuous with respect to a σ -finite nonnegative measure μ then all operator measures M_B , $B \in \mathcal{L}(\mathcal{K})$, are absolutely continuous with respect to μ (since $\mathcal{M}(X, B) \leq \mathcal{M}(X, \|B\| I_{\mathcal{K}})$ by positivity).

- (3) $\mathcal{M}(X, B) = \sum_{k=1}^{n(X)} \mathbf{A}_k(X)^* B \mathbf{A}_k(X)$ ultraweakly where $n(X) \leq \dim \mathcal{H} \dim \mathcal{K}$ and, for all $X \in \Sigma$, bounded operators $\mathbf{A}_k(X) : \mathcal{H} \rightarrow \mathcal{K}$, $1 \leq k < n(X) + 1$, are linearly independent (a minimal setwise Kraus form of \mathcal{M}).
- (4) $\mathcal{M} \in \text{Ext Ins}(\Sigma, \mathcal{K}, \mathcal{H})$ if and only if, for any decomposable operator $D = \int_{\Omega}^{\oplus} D(x) d\mu(x) \in \mathcal{L}(\mathcal{H}_{\oplus})$, the condition $Y^*(I_{\mathcal{K}} \otimes D)Y = 0$ implies $D = 0$.
- (5) $\mathcal{M} \in \text{Ext Ins}(\Sigma, \mathcal{K}, \mathcal{H})$ if and only if, for any decomposable operator $D = \int_{\Omega}^{\oplus} D(x) d\mu(x) \in \mathcal{L}(\mathcal{H}_{\oplus})$, the condition

$$\sum_{s=1}^{\dim \mathcal{K}} \int_{\Omega} \langle \varphi | \mathbf{A}^s(x)^* D(x) \mathbf{A}^s(x) \psi \rangle d\mu(x) = 0, \quad \varphi, \psi \in V_{\mathbf{h}},$$

implies $D = 0$.

Proof. (1) Let $\mathcal{M}(X, B) = W^*(B \otimes \mathbf{E}(X))W$ be a Stinespring dilation of \mathcal{M} , where $W : \mathcal{H} \rightarrow \mathcal{K} \otimes \mathcal{H}'$ is an isometry and $\mathbf{E} : \Sigma \rightarrow \mathcal{L}(\mathcal{H}')$ a spectral measure acting on a possibly nonseparable Hilbert space \mathcal{H}' (see, e.g. [10, 20, 31, 35]). Write W of the form

$$W = \sum_{m=1}^{\dim \mathcal{H} \dim \mathcal{K}} \sum_{t=1}^{\dim \mathcal{K}} |k_t \otimes \varphi_m^t\rangle \langle h_m|,$$

where $\varphi_m^t \in \mathcal{H}'$ and $\sum_{t=1}^{\dim \mathcal{K}} \langle \varphi_n^t | \varphi_m^t \rangle = \delta_{nm}$, and let $\mathcal{H}'' \subseteq \mathcal{H}'$ be a separable Hilbert space spanned by countable set of vectors φ_m^t . Denote by P the projection from \mathcal{H}' onto \mathcal{H}'' . Then

$$\mathcal{M}(X, B) = W^*(B \otimes P\mathbf{E}(X)P)W$$

where $X \mapsto P\mathbf{E}(X)P$ can be viewed as a POVM $\Sigma \rightarrow \mathcal{L}(\mathcal{H}'')$, and hence it can be diagonalized by Theorem 1. We have $\mathcal{M}(X, B) = Y^*(B \otimes \hat{\chi}_X)Y$ where $Y : \mathcal{H} \rightarrow \mathcal{K} \otimes \mathcal{H}'_{\oplus}$,

$$Y = \sum_{m=1}^{\dim \mathcal{H} \dim \mathcal{K}} \sum_{t=1}^{\dim \mathcal{K}} |k_t \otimes \psi_m^t\rangle \langle h_m|,$$

is an isometry and $\mathcal{H}'_{\oplus} = \int_{\Omega}^{\oplus} \mathcal{H}_{n'(x)} d\mu(x) \subseteq L^2(\mu, \mathcal{L})$ a direct integral Hilbert space.

Let then $\mathcal{H}_{\oplus} \subseteq \mathcal{H}'_{\oplus}$ be a closure of the linear span of vectors $\chi_X \psi_m^t$. Especially, any $\psi_m^t \in \mathcal{H}_{\oplus}$. Since \mathcal{H}_{\oplus} is a Hilbert (sub)space we may define a projection R from \mathcal{H}'_{\oplus} onto \mathcal{H}_{\oplus} . It is easy to see that R commutes with any $\hat{\chi}_X$ so that it is decomposable, $R = \int_{\Omega}^{\oplus} R(x) d\mu(x)$, by Proposition 1. Hence, \mathcal{H}_{\oplus} is a direct integral and unitarily equivalent to some $\int_{\Omega}^{\oplus} \mathcal{H}_{n(x)} d\mu(x) \subseteq L^2(\mu, \mathcal{L})$ (defined as before) so that we may set $\mathcal{H}_{\oplus} = \int_{\Omega}^{\oplus} \mathcal{H}_{n(x)} d\mu(x)$. Now Y is actually an isometry from \mathcal{H} to $\mathcal{K} \otimes \mathcal{H}_{\oplus}$. Note that (almost everywhere) $n(x) \leq n'(x) \leq \dim \mathcal{H}'' \leq \dim \mathcal{H} \dim \mathcal{K}$. Obviously vectors $(B \otimes \hat{\chi}_X)Y\psi$, $B \in \mathcal{L}(\mathcal{K})$, $X \in \Sigma$, $\psi \in \mathcal{H}$, span $\mathcal{K} \otimes \mathcal{H}_{\oplus}$ so that the above Stinespring dilation is minimal. Since

$$\begin{aligned} \langle h_n | \mathcal{M}(X, B) h_m \rangle &= \langle h_n | Y^*(B \otimes \hat{\chi}_X) Y h_m \rangle = \sum_{s,t=1}^{\dim \mathcal{K}} B_{st} \langle \psi_n^s | \chi_X \psi_m^t \rangle \\ &= \sum_{s,t=1}^{\dim \mathcal{K}} B_{st} \int_X \langle \psi_n^s(x) | \psi_m^t(x) \rangle d\mu(x) \end{aligned}$$

and the first part of the proof follows.

- (2) Pick a representative $\Omega \ni x \mapsto \psi_n^s(x) \in \mathcal{H}_{n(x)}$ from any class ψ_n^s such that

$$\sum_{s=1}^{\dim \mathcal{K}} \|\psi_n^s(x)\|^2 < \infty$$

for all $x \in \Omega$ and $n < \dim \mathcal{H} + 1$. This is possible since the set $\{\psi_n^s\}_{n,s}$ is countable and

$$\langle h_n | \mathcal{M}(X, I_{\mathcal{K}}) h_n \rangle = \sum_{s=1}^{\dim \mathcal{K}} \int_X \|\psi_n^s(x)\|^2 d\mu(x) \leq 1.$$

Then, for all $x \in \Omega$, define $d_k^s(x) := \sum_{n=1}^{\dim \mathcal{H}} \langle \psi_n^s(x) | b_k \rangle h_n \in V_{\mathbf{h}}^\times$ and $A_k(x) := \sum_{s=1}^{\dim \mathcal{K}} |k_s\rangle \langle d_k^s(x)|$ so that $A_k(x) h_n = \sum_{s=1}^{\dim \mathcal{K}} \langle b_k | \psi_n^s(x) \rangle k_s$ implying

$$\begin{aligned} \|A_k(x) h_n\|^2 &= \sum_{s=1}^{\dim \mathcal{K}} |\langle b_k | \psi_n^s(x) \rangle|^2 \\ &\leq \sum_{k=1}^{n(x)} \sum_{s=1}^{\dim \mathcal{K}} |\langle b_k | \psi_n^s(x) \rangle|^2 = \sum_{s=1}^{\dim \mathcal{K}} \|\psi_n^s(x)\|^2 < \infty \end{aligned}$$

and thus $A_k(x) \psi \in \mathcal{K}$ for all $\psi \in V_{\mathbf{h}}$. Moreover,

$$\begin{aligned} \langle \psi_n^s(x) | \psi_m^t(x) \rangle &= \sum_{k=1}^{n(x)} \langle \psi_n^s(x) | b_k \rangle \langle b_k | \psi_m^t(x) \rangle = \sum_{k=1}^{n(x)} \langle h_n | d_k^s(x) \rangle \langle d_k^t(x) | h_m \rangle \\ &= \sum_{k=1}^{n(x)} \langle h_n | A_k(x)^* k_s \rangle \langle k_t | A_k(x) h_m \rangle. \end{aligned}$$

For all $x \in \Omega$, the operators $A_k(x)$, $k < n(x) + 1$, can be chosen to be linearly independent: Indeed, suppose that there exists a set $X' \in \Sigma$ such that, for all $x \in X'$, $n(x) > 0$ and the set $\{A_k(x)\}_{k=1}^{n(x)}$ is linearly dependent. Then, for all $x \in X'$, there exists complex numbers c_k^x , $k < n(x) + 1$, such that $c_k^x \neq 0$ for finitely many k 's and $\sum_{k=1}^{n(x)} c_k^x A_k(x) = 0$. This implies that, by defining a nonzero $\varphi_x := \sum_{k=1}^{n(x)} \overline{c_k^x} b_k \in \mathcal{H}_{n(x)}$,

$$\langle \varphi_x | \psi_n^s(x) \rangle = \sum_{k=1}^{n(x)} c_k^x \langle b_k | \psi_n^s(x) \rangle = \sum_{k=1}^{n(x)} c_k^x \langle k_s | A_k(x) h_n \rangle = 0$$

for all $x \in X'$, $n < \dim \mathcal{H} + 1$, and $s < \dim \mathcal{K} + 1$. Let \mathcal{H}_x be the closure of

$$\text{lin}\{\psi_n^s(x) \mid n < \dim \mathcal{H} + 1, s < \dim \mathcal{K} + 1\}$$

in $\mathcal{H}_{n(x)}$. Since, for all $x \in X'$, the orthogonal complement \mathcal{H}_x^\perp of \mathcal{H}_x in $\mathcal{H}_{n(x)}$ is nonzero, we may choose a $\varphi \in \mathcal{H}_\oplus$ such that $0 \neq \varphi(x) \in \mathcal{H}_x^\perp$ for all $x \in X'$. But then $\langle \chi_{X'} \varphi | \chi_X \psi_n^s \rangle = 0$ for all X, n, s implying that $\chi_{X'} \varphi = 0$ by the density of the linear combinations of the vectors $\chi_X \psi_n^s$. Hence, $\mu(X') = 0$ and we may simply redefine $n(x)$ to be zero for all $x \in X'$.

Finally, $A^t(x) := \sum_{k=1}^{n(x)} |b_k\rangle \langle d_k^t(x)| = \sum_{n=1}^{\dim \mathcal{H}} |\psi_n^t(x)\rangle \langle h_n|$ is obviously an operator from $V_{\mathbf{h}}$ to $\mathcal{H}_{n(x)}$ and the last claim follows easily.

(3) The equation $\mathcal{M}(X, B) = \sum_{k=1}^{n(X)} A_k(X)^* B A_k(X)$ is just the usual Kraus decomposition [28] of a completely positive map $B \mapsto \mathcal{M}(X, B)$.

(4) From Theorem 9 and Example 12 we see that, since the unital $*$ -homomorphism is now

$$\pi : \mathcal{L}(\mathcal{K}) \otimes L^\infty(\mu) \rightarrow \mathcal{L}(\mathcal{K} \otimes \mathcal{H}_\oplus), \quad B \otimes f \mapsto B \otimes \hat{f},$$

the instrument \mathcal{M} is extreme if and only if, for all $F \in \mathcal{L}(\mathcal{K} \otimes \mathcal{H}_\oplus)$ such that $F\pi(B \otimes f) = \pi(B \otimes f)F$ for all $B \in \mathcal{L}(\mathcal{K})$, $f \in L^\infty(\mu)$, the condition $Y^*FY = 0$ implies $F = 0$. But if $F(B \otimes \hat{f}) = (B \otimes \hat{f})F$ for all B and f then $F = I_{\mathcal{K}} \otimes D$ for some decomposable $D \in \mathcal{L}(\mathcal{H}_\oplus)$ (since the commutant of the tensor product of two von Neumann algebras is the tensor product

of the commutants of the algebras in question; the commutant of $\mathcal{L}(\mathcal{K})$ is $\mathbb{C}I_{\mathcal{K}}$, see also Proposition 1). Finally, (5) follows immediately from (4). \square

We have collected the basic operators and vectors related to an instrument in the Appendix, see Remark 8. Moreover, we show there that the results of Theorem 2 do not essentially depend, e.g., on the choices of the bases \mathbf{h} and \mathbf{k} .

Remark 2. The extremality condition $\sum_{s=1}^{\dim \mathcal{K}} \int_{\Omega} \langle \varphi | \mathbf{A}^s(x)^* D(x) \mathbf{A}^s(x) \psi \rangle d\mu(x) = 0$, $\varphi, \psi \in V_{\mathbf{h}}$, of item (5) of Theorem 2 can equivalently be written in the following forms:

$$\begin{aligned} \sum_{s=1}^{\dim \mathcal{K}} \int_{\Omega} \langle \psi_n^s(x) | D(x) \psi_m^s(x) \rangle d\mu(x) &= 0, \quad 1 \leq n, m < \dim \mathcal{H} + 1, \\ \sum_{s=1}^{\dim \mathcal{K}} \int_{\Omega} \sum_{k,l=1}^{n(x)} D(x)_{kl} \langle \varphi | d_k^s(x) \rangle \langle d_l^s(x) | \psi \rangle d\mu(x) &= 0, \quad \varphi, \psi \in V_{\mathbf{h}}, \\ \int_{\Omega} \sum_{k,l=1}^{n(x)} D(x)_{kl} \langle \varphi | \mathbf{A}_k(x)^* \mathbf{A}_l(x) | \psi \rangle d\mu(x) &= 0, \quad \varphi, \psi \in V_{\mathbf{h}}, \end{aligned}$$

where $D(x)_{kl} := \langle b_k | D(x) b_l \rangle$. A mathematically elegant characterization of extreme instruments is the following: Let $\mathcal{D}(\mathcal{H}_{\oplus}) \subseteq \mathcal{L}(\mathcal{H}_{\oplus})$ be the C^* -algebra of decomposable operators $D = \int_{\Omega}^{\oplus} D(x) d\mu(x) \in \mathcal{L}(\mathcal{H}_{\oplus})$. It is the commutant of the C^* -algebra $L^{\infty}(\mu) \subseteq \mathcal{L}(\mathcal{H}_{\oplus})$ by Proposition 1. For any $\mathcal{M} \in \text{Ins}(\Sigma, \mathcal{K}, \mathcal{H})$ (with \mathcal{H}_{\oplus} and Y as in Theorem 2) define its bilinear ‘extension’

$$\overline{\mathcal{M}} : \mathcal{D}(\mathcal{H}_{\oplus}) \times \mathcal{L}(\mathcal{K}) \rightarrow \mathcal{L}(\mathcal{H}), \quad (D, B) \mapsto \overline{\mathcal{M}}(D, B) := Y^*(B \otimes D)Y$$

for which $\overline{\mathcal{M}}(\hat{\chi}_X, B) = \mathcal{M}(X, B)$. Then, $\mathcal{M} \in \text{Ext Ins}(\Sigma, \mathcal{K}, \mathcal{H})$ if and only if $D \mapsto \overline{\mathcal{M}}(D, I_{\mathcal{K}})$ is injective.

Remark 3 (Dirac formalism). Let $S = S^*$ be a (possibly unbounded) self-adjoint operator on \mathcal{H} and \mathbf{M} its spectral measure (defined on the Borel σ -algebra of \mathbb{R}). Let $d_k(x)$ be the generalized vectors of Theorem 1 associated with \mathbf{M} . As shown in [25], there exists an ON basis \mathbf{h} of \mathcal{H} such that $SV_{\mathbf{h}} \subseteq V_{\mathbf{h}}$, and if $S^{\times} : V_{\mathbf{h}}^{\times} \rightarrow V_{\mathbf{h}}^{\times}$ is the extension of S , one gets $S^{\times} d_k(x) = x d_k(x)$ for almost all x in the spectrum of S . Hence, Theorem 1 can be viewed as a generalization of *Dirac formalism* for POVMs and we may call $n(x)$ the multiplicity of a measurement outcome x .

Solutions $d_k(x)$ of the ‘eigenvalue’ equation $S^{\times} d_k(x) = x d_k(x)$ turn out to be extremely useful for determining the spectral measure \mathbf{M} of S in many practical situations. Similarly, the generalized vectors $d_k^t(x)$ of item (2) of Theorem 2 are useful, e.g., for determining \mathbf{M} -compatible instruments \mathcal{M} as we will see later. For example, if vectors $d_k^t(x)$ are related with \mathcal{M} then the generalized vectors $d_k(x)$ of the associate POVM \mathbf{M} of \mathcal{M} must satisfy

$$\sum_{k=1}^{n'(x)} |d_k(x)\rangle \langle d_k(x)| = \sum_{t=1}^{\dim \mathcal{K}} \sum_{k=1}^{n(x)} |d_k^t(x)\rangle \langle d_k^t(x)|$$

(weakly on $V_{\mathbf{h}}$) where the multiplicities $n(x)$ and $n'(x)$ are not necessarily the same.

Remark 4. It should be stressed that the direct integral Hilbert space \mathcal{H}_{\oplus} of Theorem 2 is not necessarily separable. However, when Σ is countably generated then \mathcal{H}_{\oplus} is separable (see, e.g. [6]). This holds, for instance, when Σ is the Borel σ -algebra $\mathcal{B}(\Omega)$ of a second countably topological space Ω . Hence, in the physically relevant examples, one can assume that \mathcal{H}_{\oplus} is separable. Note that in the proof of (1) of Theorem 2, one cannot assume that the spectral

measure \mathbf{E} of a Stinespring dilation acts on a separable Hilbert space. Thus, in nonseparable cases, it is questionable whether one can directly diagonalize \mathbf{E} .

3.3. Examples. In this subsection, we consider some special cases of Theorem 2. We assume that $\mathcal{M} \in \text{Ins}(\Sigma, \mathcal{K}, \mathcal{H})$ and use the notations of Theorem 2.

Example 1 (Observables: $\dim \mathcal{K} = 1$). Applying Theorem 2 to the trivial instrument $\mathcal{M} = \mathcal{M}^{\mathbf{M}}$ of a POVM \mathbf{M} , we see that Theorem 1 is a special case of Theorem 2. Indeed, now $\mathcal{K} = \mathbb{C}$, $\mathcal{K} \otimes \mathcal{H}_{\oplus} \cong \mathcal{H}_{\oplus}$, indices t and s run from 1 to $\dim \mathcal{K} = 1$, $k_1 = 1$, and $B_{11} = c \in \mathbb{C}$. It is easy to see that $\mathbf{M} \mapsto \mathcal{M}^{\mathbf{M}}$ is an affine bijection from $\text{Obs}(\Sigma, \mathcal{H})$ onto $\text{Ins}(\Sigma, \mathbb{C}, \mathcal{H})$, and that $\mathcal{M}^{\mathbf{M}} \in \text{Ext Ins}(\Sigma, \mathbb{C}, \mathcal{H})$ if and only if $\mathbf{M} \in \text{Ext Obs}(\Sigma, \mathcal{H})$.

Example 2 (Preparations: $\dim \mathcal{H} = 1$). Let $\mathcal{H} = \mathbb{C}$ (and $\mathcal{L}(\mathcal{H}) \cong \mathbb{C}$) so that indices m and n run from 1 to 1 and $h_1 = 1$, and let $\mathcal{M} \in \text{Ins}(\Sigma, \mathcal{K}, \mathbb{C})$. Drop indices n and m out from the notations. Then one has the following identifications: $\psi^s := \psi_1^s$, $d_k^s(x) = \langle \psi^s(x) | b_k \rangle \in \mathbb{C}$, $A_k(x) = a_k(x) := \sum_s \overline{d_k^s(x)} k_s \in \mathcal{K}$ (linearly independent vectors), $A^s(x) = \psi^s(x)$, and

$$\mathcal{M}(X, B) = \sum_{s,t=1}^{\dim \mathcal{K}} B_{st} \int_X \langle \psi^s(x) | \psi^t(x) \rangle d\mu(x) = \int_X \sum_{k=1}^{n(x)} \langle a_k(x) | B a_k(x) \rangle d\mu(x) = \text{tr} [\rho(X) B]$$

where

$$\rho : \Sigma \rightarrow \mathcal{T}(\mathcal{K}), \quad X \mapsto \rho(X) := \int_X \sum_{k=1}^{n(x)} |a_k(x)\rangle \langle a_k(x)| d\mu(x)$$

is a positive trace class valued operator measure for which

$$1 = \text{tr} [\rho(\Omega)] = \sum_{s=1}^{\dim \mathcal{K}} \int_{\Omega} \|\psi^s(x)\|^2 d\mu(x) = \int_{\Omega} \sum_{k=1}^{n(x)} \|a_k(x)\|^2 d\mu(x),$$

i.e. $\rho(\Omega) \in \mathcal{S}(\mathcal{K})$ is a state. By defining linearly independent unit vectors $\varphi_k(x) := a_k(x) / \|a_k(x)\|$ (when $a_k(x) \neq 0$) and $\lambda_k(x) := \|a_k(x)\|^2 \in [0, 1]$ one sees that

$$\rho(\Omega) = \int_{\Omega} \sum_{k=1}^{n(x)} \lambda_k(x) |\varphi_k(x)\rangle \langle \varphi_k(x)| d\mu(x)$$

is a (possibly uncountable or continuous) ‘convex combination’ of pure states $|\varphi_k(x)\rangle \langle \varphi_k(x)|$. Physically, $\rho(\Omega)$ could be associated with some preparation procedure which produces convex combinations of pure states. For example, a radiation source emits pure states $|\varphi_k(x)\rangle \langle \varphi_k(x)|$ randomly so that the output state must be assumed to be mixed with weights $\lambda_k(x)$.

From Theorem 1 one sees that, any positive operator measure $\mathbf{M} : \Sigma \rightarrow \mathcal{L}(\mathcal{K})$ for which $\mathbf{M}(\Omega) \in \mathcal{T}(\mathcal{K})$ and $\text{tr} [\mathbf{M}(\Omega)] = 1$, is of the form

$$\mathbf{M}(X) = \sum_{t,s=1}^{\dim \mathcal{K}} \int_X \langle \psi_t(x) | \psi_s(x) \rangle d\mu(x) |k_t\rangle \langle k_s| \in \mathcal{T}(\mathcal{K})$$

where $\text{tr} [\mathbf{M}(\Omega)] = \sum_{s=1}^{\dim \mathcal{K}} \int_{\Omega} \|\psi_s(x)\|^2 d\mu(x) = 1$. Define then a positive operator measure $\rho' : \Sigma \rightarrow \mathcal{T}(\mathcal{K})$ by

$$\rho'(X) := \sum_{t,s=1}^{\dim \mathcal{K}} \int_X \langle \psi_s(x) | \psi_t(x) \rangle d\mu(x) |k_t\rangle \langle k_s|,$$

and an instrument $\mathcal{M}' \in \text{Ins}(\Sigma, \mathcal{K}, \mathbb{C})$ by $\mathcal{M}'(X, B) := \text{tr} [\rho'(X) B]$. The correspondence $\mathbf{M} \mapsto \mathcal{M}'$ is an affine bijection.

Note that $\mathcal{M} \in \text{Ext Ins}(\Sigma, \mathcal{K}, \mathbb{C})$ if and only if $\sum_{s=1}^{\dim \mathcal{K}} \int_{\Omega} \langle \psi^s(x) | D(x) \psi^s(x) \rangle d\mu(x)$ implies $D = 0$. We have two special cases:

- a) Also $\mathcal{K} = \mathbb{C}$. Then $\text{Ins}(\Sigma, \mathbb{C}, \mathbb{C})$ is just a convex set of probability measures (classical states) $\mu : \Sigma \rightarrow [0, 1]$ (for which $\psi^1(x) \equiv 1$) and we see that $\mu \in \text{Ext Ins}(\Sigma, \mathbb{C}, \mathbb{C})$ if and only if, for all $d(x) \in L^\infty(\mu)$, the condition $\int_{\Omega} d(x) d\mu(x) = 0$ implies $d = 0$, if and only if $\mu(\Sigma) \in \{0, 1\}$.
- b) Let $\Omega = \{0\}$ and consider $\mathcal{M} \in \text{Ins}(2^{\{0\}}, \mathcal{K}, \mathbb{C})$. Then the corresponding $\rho(\{0\}) \in \mathcal{T}(\mathcal{K})$ is a state and $\text{Ins}(2^{\{0\}}, \mathcal{K}, \mathbb{C})$ is a convex set of states on \mathcal{K} . One sees immediately that \mathcal{M} is extreme if and only if $\rho(\{0\})$ is pure, i.e. $\rho(\{0\}) = |a_1(\{0\})\rangle\langle a_1(\{0\})|$.

Example 3 (The discrete case). Let $\mathcal{M} \in \text{Ins}(\Sigma, \mathcal{K}, \mathcal{H})$ and $\mathbf{M}_{I_{\mathcal{K}}}$ the corresponding POVM. Suppose that there exists a finite or countably infinite set $\overline{X} = \{x_i\}_{i=1}^N \subseteq \Omega$, $N \in \mathbb{N}_{\infty}$, such that $\{x_i\} \in \Sigma$, $\mathbf{M}_{I_{\mathcal{K}}}(\{x_i\}) > 0$ for all i and $\mathbf{M}_{I_{\mathcal{K}}}(\overline{X}) = I_{\mathcal{H}}$. Then μ can be chosen to be such that $\mu(\{x_i\}) = 1$ for all i and $\mu(\Omega \setminus \overline{X}) = 0$. It follows that

$$\mathcal{M}(X, B) = \sum_{\substack{1 \leq i < N+1 \\ x_i \in X}} T_i(B)$$

where any $T_i : \mathcal{L}(\mathcal{K}) \rightarrow \mathcal{L}(\mathcal{H})$, $B \mapsto T_i(B) := \mathcal{M}(\{x_i\}, B)$ is a CP operation and all integrals in Theorem 2 reduce to sums. By replacing each x_i by i in the notations [e.g. $\mathbf{A}_k(i) := \mathbf{A}_k(x_i) = \mathbf{A}_k(\{x_i\})$] we have $\mathcal{H}_{\oplus} = \bigoplus_{i=1}^N \mathcal{H}_{n(i)}$, $\psi_m^t(i) \in \mathcal{H}_{n(i)}$, $d_k^t(i) \in \mathcal{H}$, and the extended operators $\mathbf{A}_k(i) : \mathcal{H} \rightarrow \mathcal{K}$ and $\mathbf{A}^t(i) : \mathcal{H} \rightarrow \mathcal{H}_{n(i)}$ are bounded. Hence, we have $\sum_{t,i} \langle \psi_n^t(i) | \psi_m^t(i) \rangle = \delta_{nm}$ and (weakly)

$$\begin{aligned} T_i(B) &= \sum_{n,m=1}^{\dim \mathcal{H}} \sum_{s,t=1}^{\dim \mathcal{K}} B_{st} \langle \psi_n^s(i) | \psi_m^t(i) \rangle |h_n\rangle\langle h_m| = \sum_{s,t=1}^{\dim \mathcal{K}} B_{st} \sum_{k=1}^{n(i)} |d_k^s(i)\rangle\langle d_k^t(i)| \\ &= \sum_{k=1}^{n(i)} \mathbf{A}_k(i)^* B \mathbf{A}_k(i) = \sum_{s,t=1}^{\dim \mathcal{K}} B_{st} \mathbf{A}^s(i)^* \mathbf{A}^t(i) \end{aligned}$$

weakly. We say that $n(i)$ is the *rank* of T_i . Immediately one gets a generalization of Theorem 5 of [9]: $\mathcal{M} \in \text{Ext Ins}(\Sigma, \mathcal{K}, \mathcal{H})$ if and only if $\sum_{i,k,l} D(i)_{kl} \mathbf{A}_k(i)^* \mathbf{A}_l(i) = 0$ implies $D = 0$ (where $D = \bigoplus_{i=1}^N D(i)$ is bounded). In the next example, we concentrate on the channel case $N = 1$ and drop (i) out from the notations.

Example 4 (Channels). Let $T : \mathcal{L}(\mathcal{K}) \rightarrow \mathcal{L}(\mathcal{H})$ be a CP channel and $T(B) = \sum_{k=1}^n \mathbf{A}_k^* B \mathbf{A}_k$, $B \in \mathcal{L}(\mathcal{K})$, its minimal Kraus decomposition (where the bounded operators $\mathbf{A}_k : \mathcal{H} \rightarrow \mathcal{K}$ are linearly independent and $n \leq \dim \mathcal{H} \dim \mathcal{K}$ is minimal). From Example 3 we get that the channel T is an extreme point of the convex set of CP channels $\mathcal{L}(\mathcal{K}) \rightarrow \mathcal{L}(\mathcal{H})$ if and only if, for any $n \times n$ -complex matrix (D_{kl}) with the finite operator norm the condition $\sum_{k,l=1}^n D_{kl} \mathbf{A}_k^* \mathbf{A}_l = 0$ (weakly) implies $D_{kl} \equiv 0$. This has also been proved in [36, Theorem 2.4] (see also [8, Theorem 5] and an alternative formulation [22, Proposition 1]). Especially, when the rank n of T is finite, then T is extreme if and only if the operators $\mathbf{A}_k^* \mathbf{A}_l$ are linearly independent. For example, if the rank $n = 1$, i.e. $T(B) = \mathbf{A}_1^* B \mathbf{A}_1$, then T is extreme.

Example 5 (The finite dimensional case). Let $\mathcal{M} \in \text{Ext Ins}(\Sigma, \mathcal{K}, \mathcal{H})$ with the isometry Y of Theorem 2, and let \mathbf{M} be the associate POVM of \mathcal{M} . For any $f \in L^\infty(\mu)$, the condition

$$(1) \quad \int_{\Omega} f(x) d\mathbf{M}(x) = Y^*(I_{\mathcal{K}} \otimes \hat{f})Y = 0$$

implies $f = 0$. Let $X_1, \dots, X_N \in \Sigma$ be disjoint sets such that $\mathbf{M}(X_i) \neq 0$. Then the effects $\mathbf{M}(X_1), \dots, \mathbf{M}(X_N)$ are linearly independent which can be seen by substituting $f = \sum_{i=1}^N c_i \hat{\chi}_{X_i}$

into equation (1):

$$\sum_{i=1}^N c_i \mathbf{M}(X_i) = 0 \quad \text{implies} \quad c_1 = c_2 = \dots = c_N = 0.$$

From this fact follows that, if \mathcal{M} is extremal then there are at most $(\dim \mathcal{H})^2$ disjoint sets X_i such that $\mathbf{M}(X_i) \neq 0$. Since $\mathbf{M}(X) = 0$ implies $\mathcal{M}(X, B) = 0$ for all $B \in \mathcal{L}(\mathcal{K})$, an extremal \mathcal{M} is concentrated on the set $\cup_{i=1}^N X_i$, $N \leq (\dim \mathcal{H})^2$, but it does not necessarily follow that \mathbf{M} is discrete even when $\dim \mathcal{H} < \infty$ [14]. By adding topological assumptions we get (for the proof, see [14]):

Proposition 2. *Suppose that $(\Omega, \Sigma = \mathcal{B}(\Omega))$ is a second countable Hausdorff space, $\dim \mathcal{H} < \infty$, and $\mathcal{M} \in \text{Ext Ins}(\Sigma, \mathcal{K}, \mathcal{H})$. Then \mathcal{M} is concentrated on a finite set, i.e.,*

$$\mathcal{M}(X, B) = \sum_{i=1}^N \chi_X(x_i) T_i(B), \quad X \in \Sigma, B \in \mathcal{L}(\mathcal{K}),$$

for some finite number $N \leq (\dim \mathcal{H})^2$ of elements $x_1, \dots, x_N \in \Omega$ and CP operations $T_i : \mathcal{L}(\mathcal{K}) \rightarrow \mathcal{L}(\mathcal{H})$.

Example 6 (Multi-instruments). Quantum measurement and information processes can be viewed as combinations of the basic building blocks, elementary instruments, introduced in the preceding examples. For example, first one could start from a trivial (or ‘classical’) Hilbert space \mathbb{C} and prepare a state in \mathcal{K} . Then one could process the state by using channels and instruments. Finally, the process ends in a measurement of a POVM, the final Hilbert space being the classical space \mathbb{C} again. This kind of processes can be viewed as the following combinations of (Heisenberg) instruments $\mathcal{M}_i \in \text{Ins}(\Sigma_i, \mathcal{K}_i, \mathcal{H}_i)$, $i = 1, 2, \dots, N \in \mathbb{N}_+$,

$$\mathcal{M}_1 \left(X_1, \mathcal{M}_2 \left(X_2, \mathcal{M}_3 \left(X_3, \dots \mathcal{M}_N(X_N, B_N) \right) \right) \right) \in \mathcal{L}(\mathcal{H}_1), \quad X_i \in \Sigma_i, B_N \in \mathcal{L}(\mathcal{K}_N)$$

so that one must have $\mathcal{K}_i = \mathcal{H}_{i+1}$ for all $i = 1, \dots, N-1$. We denote the above combination by $\overline{\mathcal{M}}((X_1, X_2, \dots, X_N), B_N)$ and say that the corresponding map

$$\overline{\mathcal{M}} : \Sigma_1 \times \Sigma_2 \times \dots \times \Sigma_N \times \mathcal{L}(\mathcal{K}_N) \rightarrow \mathcal{L}(\mathcal{H}_1)$$

is the *multi-instrument* generated by the instruments $\mathcal{M}_1, \dots, \mathcal{M}_N$. By induction, it suffices to consider the case $N = 2$. Next we consider an important example (see a recent paper [5]).

Assume that we measure POVMs \mathbf{M}_1 and \mathbf{M}_2 (of the same Hilbert space \mathcal{H}) by performing their measurements sequentially (first \mathbf{M}_1 and then \mathbf{M}_2). This leads to the bi-instrument $\overline{\mathcal{M}}$ defined by

$$\overline{\mathcal{M}}(X_1 \times X_2, B_2) := \mathcal{M}_1(X_1, \mathcal{M}_2(X_2, B_2)).$$

Usually $\overline{\mathcal{M}}$ extends to an instrument on a product σ -algebra [11, Theorem 2]. It defines a sequential joint observable \mathbf{M}_{12} whose margins are POVMs

$$X_1 \mapsto \mathbf{M}_{12}(X_1 \times \Omega_2, I_{\mathcal{H}}) = \mathbf{M}_1(X_1), \quad X_2 \mapsto \mathbf{M}_{12}(\Omega_1 \times X_2, I_{\mathcal{H}}) = \mathcal{M}_1(\Omega_1, \mathbf{M}_2(X_2))$$

where the channel $\mathcal{M}_1(\Omega_1, \bullet)$ operates to \mathbf{M}_2 , that is, the first measurement disturbs the subsequent one. Finally, we note that, if $\{\psi_n^s\}$ and $\{\psi'_n{}^s\}$ are the structure vectors of \mathcal{M}_1 and \mathcal{M}_2 of Theorem 2, respectively, the (not necessarily minimal) structure vectors $\{\psi''_n{}^s\}$ of $\overline{\mathcal{M}}$ can be easily calculated:

$$\psi''_n{}^s = \sum_a \psi_n^a \otimes \psi'_a{}^s.$$

4. M-COMPATIBLE INSTRUMENTS

Let $\mathcal{M} \in \text{Ins}(\Sigma, \mathcal{H}, \mathcal{K})$ with an isometry $Y : \mathcal{H} \rightarrow \mathcal{K} \otimes \mathcal{H}_\oplus$ and structure vectors $\psi_n^s \in L^2(\mu, \mathcal{L})$ given by (1) of Theorem 2. Let $\{\mathbf{M}_B\}_{B \in \mathcal{L}(\mathcal{K})}$ be the family of operator measures associated to \mathcal{M} . Since any B can be decomposed into positive parts one sees that \mathbf{M}_B can also be decomposed into a sum of positive operator measures (compare to [26]). Suppose then that B is positive with the square root operator \sqrt{B} . Now

$$\mathbf{M}_B(X) = \mathcal{M}(X, B) = Y^*(B \otimes \hat{\chi}_X)Y = Y_B^*(I_{\mathcal{K}} \otimes \hat{\chi}_X)Y_B$$

where

$$Y_B := (\sqrt{B} \otimes I_{\mathcal{H}_\oplus})Y$$

is an operator from \mathcal{H} to a Hilbert space $\mathcal{H}^B \subseteq \mathcal{K} \otimes \mathcal{H}_\oplus$ defined as the closure of the linear combinations of vectors $(\sqrt{B} \otimes \hat{\chi}_X)Y\psi$, $X \in \Sigma$, $\psi \in \mathcal{H}$. Trivially, vectors $(I_{\mathcal{K}} \otimes \hat{\chi}_X)Y_B\psi$ span \mathcal{H}^B so that we have obtained a minimal Naimark dilation for \mathbf{M}_B . (By Theorem 1, \mathcal{H}^B is unitarily equivalent with a direct integral Hilbert space $\mathcal{H}_\oplus^B \subseteq L^2(\mu, \mathcal{L})$ where \mathbf{M}_B is diagonalized but we do not need this fact here.) By defining the structure vectors of \mathbf{M}_B ,

$$\psi_n^B := Y_B h_n = (\sqrt{B} \otimes I_{\mathcal{H}_\oplus})Y h_n = \sum_{s=1}^{\dim \mathcal{K}} (\sqrt{B} k_s) \otimes \psi_n^s$$

one gets

$$\begin{aligned} \mathbf{M}_B(X) &= \sum_{n,m=1}^{\dim \mathcal{H}} \int_X \langle \psi_n^B(x) | \psi_m^B(x) \rangle d\mu(x) |h_n\rangle \langle h_m| \\ &= \sum_{n,m=1}^{\dim \mathcal{H}} \sum_{s,t=1}^{\dim \mathcal{K}} B_{st} \int_X \langle \psi_n^s(x) | \psi_m^t(x) \rangle d\mu(x) |h_n\rangle \langle h_m| = \mathcal{M}(X, B). \end{aligned}$$

When $B = I_{\mathcal{K}}$, we see that the structure vectors $\psi_n^{I_{\mathcal{K}}}$ of the associate observable $\mathbf{M}_{I_{\mathcal{K}}}$ of \mathcal{M} are

$$\psi_n^{I_{\mathcal{K}}} = Y h_n = \sum_{s=1}^{\dim \mathcal{K}} k_s \otimes \psi_n^s.$$

If B is not positive then one can collect the structure vectors of its positive parts into a single structure vector as in Remark 5.8 of [26].

Let then $\mathbf{M} \in \text{Obs}(\Sigma, \mathcal{H})$ be a POVM with ON vectors $\psi_n \in \mathcal{H}_\oplus$ of Theorem 1 and try to find an *M-compatible instrument* $\mathcal{M} \in \text{Ins}(\Sigma, \mathcal{H}, \mathcal{K})$, i.e. $\mathcal{M}(X, I_{\mathcal{K}}) \equiv \mathbf{M}(X)$, with vectors ψ_n^s as in Theorem 2. Clearly, one must ‘solve’ the equation

$$(2) \quad \sum_{s=1}^{\dim \mathcal{K}} \langle \psi_n^s(x) | \psi_m^s(x) \rangle \equiv \langle \psi_n(x) | \psi_m(x) \rangle$$

for unknown vectors ψ_n^s . The next theorem characterizes completely M-compatible instruments:

Theorem 3. *Let $\mathbf{M} \in \text{Obs}(\Sigma, \mathcal{H})$ with the structure vectors $\psi_m \in \mathcal{H}_\oplus = \int_\Omega^\oplus \mathcal{H}_{n(x)} d\mu(x)$ and the isometry Y of Theorem 1 associated with the diagonal minimal Naimark dilation, and let $\mathcal{M} \in \text{Ins}(\Sigma, \mathcal{K}, \mathcal{H})$. Then $\mathcal{M}(X, I_{\mathcal{K}}) \equiv \mathbf{M}(X)$ if and only if there exists a decomposable CP channel $T : \mathcal{L}(\mathcal{K}) \rightarrow \mathcal{L}(\mathcal{H}_\oplus)$, $B \mapsto T(B) = \int_\Omega^\oplus T_x(B) d\mu(x)$, where $T_x : \mathcal{L}(\mathcal{K}) \rightarrow \mathcal{L}(\mathcal{H}_{n(x)})$ are CP channels for μ -almost all $x \in \Omega$, such that*

$$\mathcal{M}(X, B) \equiv Y^* T(B) \hat{\chi}_X Y = \sum_{n,m=1}^{\dim \mathcal{H}} \int_X \langle \psi_n(x) | T_x(B) \psi_m(x) \rangle d\mu(x) |h_n\rangle \langle h_m|.$$

Any channel T_x above can be chosen to be of the form

$$T_x(B) = C_x^*(B \otimes I_{\mathcal{H}'_{n'(x)}})C_x, \quad B \in \mathcal{L}(\mathcal{K}),$$

where $C_x : \mathcal{H}_{n(x)} \rightarrow \mathcal{K} \otimes \mathcal{H}'_{n'(x)}$ is an isometry and $\int_{\Omega}^{\oplus} \mathcal{H}'_{n'(x)} d\mu(x)$ is the direct integral Hilbert space of Theorem 2 associated to \mathcal{M} . Especially, one must have $\dim \mathcal{H}_{n(x)} \leq \dim \mathcal{K} \dim \mathcal{H}'_{n'(x)}$ for μ -almost all $x \in \Omega$.

Proof. Suppose that there exists an $\mathcal{M} \in \text{Ins}(\Sigma, \mathcal{K}, \mathcal{H})$ such that $\mathcal{M}(X, I_{\mathcal{K}}) \equiv \mathbf{M}(X)$, and let $Y : \mathcal{H} \rightarrow \mathcal{H}_{\oplus}$ and $Y' : \mathcal{H} \rightarrow \mathcal{K} \otimes \mathcal{H}'_{\oplus}$ be the isometries of Theorems 1 and 2, that is, $\mathcal{M}(X, B) \equiv Y'^*(B \otimes \hat{\chi}_X)Y'$ and $\mathbf{M}(X) \equiv Y^*\hat{\chi}_X Y \equiv Y'^*(I_{\mathcal{K}} \otimes \hat{\chi}_X)Y'$. Define

$$C \left(\sum_{i=1}^n \hat{\chi}_{X_i} Y \eta_i \right) := \sum_{i=1}^n (I_{\mathcal{K}} \otimes \hat{\chi}_{X_i}) Y' \eta_i \in \mathcal{K} \otimes \mathcal{H}'_{\oplus}$$

for all $X_i \in \Sigma$ and $\eta_i \in \mathcal{H}$ where $i = 1, 2, \dots, n \in \mathbb{N}_+$. Since

$$\begin{aligned} \left\| C \left(\sum_i \hat{\chi}_{X_i} Y \eta_i \right) \right\|^2 &= \left\langle \sum_i (I_{\mathcal{K}} \otimes \hat{\chi}_{X_i}) Y' \eta_i \middle| \sum_j (I_{\mathcal{K}} \otimes \hat{\chi}_{X_j}) Y' \eta_j \right\rangle \\ &= \sum_{i,j} \langle \eta_i | Y'^*(I_{\mathcal{K}} \otimes \hat{\chi}_{X_i \cap X_j}) Y' \eta_j \rangle = \sum_{i,j} \langle \eta_i | Y^* \hat{\chi}_{X_i \cap X_j} Y \eta_j \rangle \\ &= \left\langle \sum_i \hat{\chi}_{X_i} Y \eta_i \middle| \sum_j \hat{\chi}_{X_j} Y \eta_j \right\rangle = \left\| \sum_i \hat{\chi}_{X_i} Y \eta_i \right\|^2 \end{aligned}$$

it follows that $C : \mathcal{H}_{\oplus} \rightarrow \mathcal{K} \otimes \mathcal{H}'_{\oplus}$ is well-defined linear isometry, $C^*C = I_{\mathcal{H}_{\oplus}}$. Moreover, $(I_{\mathcal{K}} \otimes \hat{\chi}_X)C = C\hat{\chi}_X$ for all $X \in \Sigma$ so that C is decomposable by Proposition 1, that is, $C = \int_{\Omega}^{\oplus} C_x d\mu(x)$ where the operators $C_x : \mathcal{H}_{n(x)} \rightarrow \mathcal{K} \otimes \mathcal{H}'_{n'(x)}$ are isometries for μ -almost all $x \in \Omega$; here we have denoted $\mathcal{H}'_{\oplus} = \int_{\Omega}^{\oplus} \mathcal{H}'_{n'(x)} d\mu(x)$. Since $CY = Y'$ and $(I_{\mathcal{K}} \otimes \hat{\chi}_X)C = C\hat{\chi}_X$ one sees that

$$\mathcal{M}(X, B) \equiv Y'^*(B \otimes \hat{\chi}_X)Y' = Y^*C^*(B \otimes I_{\mathcal{H}_{\oplus}})(I_{\mathcal{K}} \otimes \hat{\chi}_X)CY = Y^*T(B)\hat{\chi}_X Y$$

where $T : \mathcal{L}(\mathcal{K}) \rightarrow \mathcal{L}(\mathcal{H}_{\oplus})$, $B \mapsto T(B) := C^*(B \otimes I_{\mathcal{H}_{\oplus}})C$ is obviously a CP channel and, for all $B \in \mathcal{L}(\mathcal{K})$, $T(B)$ is decomposable, $T(B) = \int_{\Omega}^{\oplus} T_x(B) d\mu(x)$, where $T_x(B) = C_x^*(B \otimes I_{\mathcal{H}'_{n'(x)}})C_x$ for μ -almost all $x \in \Omega$. Let $\Omega_C \subseteq \Omega$ be a μ -measurable set such that C_x is an isometry for all $x \in \Omega_C$ and $\mu(\Omega \setminus \Omega_C) = 0$. Then, for each $x \in \Omega_C$, the map $B \mapsto T_x(B) = C_x^*(B \otimes I_{\mathcal{H}'_{n'(x)}})C_x$ is clearly a CP channel. The converse claim is trivial and the proof is complete. \square

Example 7 (Nuclear instruments and EB channels). If one chooses $T_x(B) = \text{tr}[\sigma_x B] I_{\mathcal{H}_{n(x)}}$, $B \in \mathcal{L}(\mathcal{K})$, where $\{\sigma_x\}_{x \in \Omega} \subseteq \mathcal{S}(\mathcal{K})$ is a (μ -measurable) family of states, one gets the *nuclear instrument*

$$\mathcal{M}(X, B) = \int_X \text{tr}[\sigma_x B] d\mathbf{M}(x), \quad X \in \Sigma, B \in \mathcal{L}(\mathcal{K}),$$

introduced by Ozawa [32]. Its predual instrument is

$$\mathcal{M}_*(X, \rho) = \int_X \sigma_x \text{tr}[\rho \mathbf{M}(dx)], \quad X \in \Sigma, \rho \in \mathcal{T}(\mathcal{H})$$

and the associate (predual) channel

$$\mathcal{M}_*(\Omega, \rho) = \int_{\Omega} \sigma_x \text{tr}[\rho \mathbf{M}(dx)], \quad X \in \Sigma, \rho \in \mathcal{T}(\mathcal{H}).$$

is *entanglement-breaking (EB)* [24, 23, 21, 18]. Following [23] one sees that a *CP channel is EB if and only if it is the associate channel of a (nonunique) nuclear instrument*. It is easy to see that an \mathbf{M} -compatible instrument is nuclear if and only if its structure vectors can be chosen to be decomposable, that is, $\psi_n^s(x) = \eta^s(x) \otimes \psi_n(x)$ where $\eta^s(x) \in \mathcal{K}$, $\sum_{s=1}^{\dim \mathcal{K}} \|\eta^s(x)\|^2 = 1$, and vectors $\psi_n(x)$ are the structure vectors of \mathbf{M} .

For example, the instrument of Davies and Lewis [11, Theorem 1] is a special case of nuclear instruments so that, for any POVM \mathbf{M} , there exists a (nontrivial) nuclear instrument implementing \mathbf{M} . Later we show that any \mathbf{M} -compatible instrument is nuclear if \mathbf{M} is of rank 1 (thus generalizing Corollary 1 of [17]).

Let $\mathbf{M} \in \text{Obs}(\Sigma, \mathcal{H})$, $\mathcal{M} \in \text{Ins}(\Sigma, \mathcal{K}, \mathcal{H})$, and assume that $\mathcal{M}(X, I_{\mathcal{K}}) \equiv \mathbf{M}(X)$. Let $\mathcal{H}_{\oplus} = \int_{\Omega}^{\oplus} \mathcal{H}_{n(x)} d\mu(x) \subseteq L^2(\mu, \mathcal{L})$ and $\mathcal{H}'_{\oplus} = \int_{\Omega}^{\oplus} \mathcal{H}_{n'(x)} d\mu(x) \subseteq L^2(\mu, \mathcal{L})$ be the direct integral Hilbert spaces of Theorems 1 and 2, and recall the notations and definitions of Subsection 3.1 and Remark 8 for \mathbf{M} and \mathcal{M} , respectively:

$$\begin{aligned} \mathbf{M}(X) &\supseteq \int_X A^1(x)^* A^1(x) d\mu(x) = \int_X \sum_{k=1}^{n(x)} A_k(x)^* A_k(x) d\mu(x), \\ A^1(x) &:= \sum_{k=1}^{n(x)} |b_k\rangle \langle d_k(x)| = \sum_{n=1}^{\dim \mathcal{H}} |\psi_n(x)\rangle \langle h_n|, \quad A_k(x) := |1\rangle \langle d_k(x)|, \\ \mathcal{M}(X, I_{\mathcal{K}}) &\supseteq \int_X \sum_{k=1}^{n'(x)} A'_k(x)^* B A'_k(x) d\mu(x), \\ A'_k(x) &:= \sum_{s=1}^{\dim \mathcal{K}} \sum_{n=1}^{\dim \mathcal{H}} \langle b_k | \psi_n^s(x) \rangle |k_s\rangle \langle h_n| = \sum_{s=1}^{\dim \mathcal{K}} |k_s\rangle \langle d_k^s(x)|. \end{aligned}$$

Let T be the decomposable CP channel of Theorem 3. Immediately we see that $\langle \psi_n^s(x) | \psi_m^t(x) \rangle = \langle \psi_n(x) | T_x(|s\rangle \langle t|) \psi_m(x) \rangle$ and (2) holds. Moreover, one obtains a *pointwise Kraus decomposition* for T :

$$\begin{aligned} \mathcal{M}(X, I_{\mathcal{K}}) &\supseteq \int_X \sum_{k=1}^{n'(x)} A'_k(x)^* B A'_k(x) d\mu(x) = \int_X A^1(x)^* \left[\sum_{k=1}^{n'(x)} A_k^T(x)^* B A_k^T(x) \right] A^1(x) d\mu(x) \\ T_x(B) &= C_x^*(B \otimes I_{\mathcal{H}_{n'(x)}}) C_x = \sum_{k=1}^{n'(x)} C_x^*(B \otimes |b_k\rangle \langle b_k|) C_x = \sum_{k=1}^{n'(x)} A_k^T(x)^* B A_k^T(x), \end{aligned}$$

where (when C_x is an isometry)

$$A_k^T(x) := \sum_{s=1}^{\dim \mathcal{K}} |k_s\rangle \langle k_s \otimes b_k| C_x = \sum_{s=1}^{\dim \mathcal{K}} |k_s\rangle \langle D_k^s(x)|$$

is a bounded operator from $\mathcal{H}_{n(x)}$ to \mathcal{K} and the generalized vectors $D_k^s(x) := C_x^*(k_s \otimes b_k)$ of the channel T_x are now elements of the Hilbert space $\mathcal{H}_{n(x)}$. Note also that then $\sum_k A_k^T(x)^* A_k^T(x) = I_{\mathcal{H}_{n(x)}}$. Since $C_x \psi_n(x) = C_x(Y h_n)(x) = (Y' h_n)(x) = \sum_s k_s \otimes \psi_n^s(x)$ we get the following

relations:

$$\begin{aligned} A'_k(x) &= A_k^T(x)A^1(x), & A_k^T(x)\psi_n(x) &= \sum_{s=1}^{\dim \mathcal{K}} \langle b_k | \psi_n^s(x) \rangle k_s, \\ d_k^s(x) &= \sum_{l=1}^{n(x)} \langle A_k^T(x)b_l | k_s \rangle d_l(x) = \sum_{l=1}^{n(x)} \langle C_x b_l | k_s \otimes b_k \rangle d_l(x) = \sum_{l=1}^{n(x)} \langle b_l | D_k^s(x) \rangle d_l(x). \end{aligned}$$

The last equation gives us a relation between the generalized vectors $d_k^s(x)$, $D_k^s(x)$, and $d_k(x)$ of \mathcal{M} , T , and \mathbf{M} , respectively. In addition, it provides an *effective tool for determining \mathbf{M} -compatible instruments*: If the generalized vectors $d_l(x)$ of \mathbf{M} are given, take a (measurable collection of) complex ‘matrices’ $(c_{l,sk}^x)$ such that the orthogonality condition

$$\sum_{s,k} c_{l,sk}^x \overline{c_{l',sk}^x} = \delta_{ll'}.$$

is satisfied. Then the instrument $\mathcal{M} \in \text{Ins}(\Sigma, \mathcal{K}, \mathcal{H})$ defined by the generalized vectors

$$d_k^s(x) := \sum_{l=1}^{n(x)} c_{l,sk}^x d_l(x)$$

is \mathbf{M} -compatible. Especially, if $n(x) = 1$ one sees that $d_k^s(x) = c_{1,sk}^x d_1(x)$ where $c_{1,sk}^x \in \mathbb{C}$ and $\sum_{s,k} |c_{1,sk}^x|^2 = 1$.

Corollary 1. *Let $\mathbf{M} \in \text{Obs}(\Sigma, \mathcal{H})$ and $\mathcal{M} \in \text{Ins}(\Sigma, \mathcal{K}, \mathcal{H})$. If $\mathcal{M}(X, I_{\mathcal{K}}) \equiv \mathbf{M}(X)$ then there exist CP channels $\Phi^X : \mathcal{L}(\mathcal{K}) \rightarrow \mathcal{L}(\mathcal{H})$, $X \in \Sigma$, such that*

$$\mathcal{M}(X, B) = \sqrt{\mathbf{M}(X)} \Phi^X(B) \sqrt{\mathbf{M}(X)}, \quad X \in \Sigma, \quad B \in \mathcal{L}(\mathcal{K}).$$

Proof. Assume that $\mathcal{M}(X, I_{\mathcal{K}}) \equiv \mathbf{M}(X)$ and let T be the channel of Theorem 3 so that $\mathcal{M}(X, B) \equiv Y^* T(B) \hat{\chi}_X Y$. Fix $X \in \Sigma$ and define, for all $\eta \in \mathcal{H}$,

$$E_X \left(\sqrt{\mathbf{M}(X)} \eta \right) := \hat{\chi}_X Y \eta$$

for which

$$\left\| E_X \left(\sqrt{\mathbf{M}(X)} \eta \right) \right\|^2 = \langle \hat{\chi}_X Y \eta | \hat{\chi}_X Y \eta \rangle = \langle \eta | Y^* \hat{\chi}_X Y \eta \rangle = \langle \eta | \mathbf{M}(X) \eta \rangle = \left\| \sqrt{\mathbf{M}(X)} \eta \right\|^2,$$

so that E_X is an isometry from $\sqrt{\mathbf{M}(X)} \mathcal{H}$ to \mathcal{H}_{\oplus} . Extend E_X to some isometry $\overline{E}_X : \mathcal{H} \rightarrow \mathcal{H}_{\oplus}$, $\overline{E}_X^* \overline{E}_X = I_{\mathcal{H}}$ (which clearly exists since $Y : \mathcal{H} \rightarrow \mathcal{H}_{\oplus}$ is isometric). Define a channel $\Phi^X : \mathcal{L}(\mathcal{K}) \rightarrow \mathcal{L}(\mathcal{H})$, $B \mapsto \Phi^X(B) := \overline{E}_X^* T(B) \overline{E}_X$ for which

$$\left\langle \eta \left| \sqrt{\mathbf{M}(X)} \Phi^X(B) \sqrt{\mathbf{M}(X)} \eta \right. \right\rangle = \langle \hat{\chi}_X Y \eta | T(B) \hat{\chi}_X Y \eta \rangle = \langle \eta | \mathcal{M}(X, B) \eta \rangle$$

for all $\eta \in \mathcal{H}$ and $B \in \mathcal{L}(\mathcal{K})$. Hence, the corollary follows. \square

Remark 5. In the context of Theorem 3 and Corollary 1, we note the following facts: Let $\mathcal{M} \in \text{Ins}(\Sigma, \mathcal{K}, \mathcal{H})$ be an *arbitrary* instrument and $\mathbf{M}(X) = \mathcal{M}(X, I_{\mathcal{K}})$ its associate observable with the minimal Naimark isometry $Y : \mathcal{H} \rightarrow \mathcal{H}_{\oplus}$ of Theorem 1.

- (1) There exists a CP channel T such that $\mathcal{M}(X, B) \equiv Y^* \overline{\mathcal{M}}(X, B) Y$ where

$$\overline{\mathcal{M}} : \Sigma \times \mathcal{L}(\mathcal{K}) \rightarrow \mathcal{L}(\mathcal{H}_{\oplus}), \quad (X, B) \mapsto \overline{\mathcal{M}}(X, B) := T(B) \hat{\chi}_X$$

is an instrument with the canonical spectral measure as its associate observable and T as its associate channel.

(2) If \mathbf{M} is projection valued then $YY^* = I_{\mathcal{H}_\oplus}$, $Y\mathcal{H} = \mathcal{H}_\oplus$, and

$$\begin{aligned}\mathcal{M}(X, B) &\equiv Y^*T(B)YY^*\hat{\chi}_X Y = T_Y(B)\mathbf{M}(X) = \mathbf{M}(X)T_Y(B)\mathbf{M}(X) \\ &= \sqrt{\mathbf{M}(X)}T_Y(B)\sqrt{\mathbf{M}(X)}\end{aligned}$$

where $T_Y := Y^*TY : \mathcal{L}(\mathcal{K}) \rightarrow \mathcal{L}(\mathcal{H})$ is a CP channel commuting with \mathbf{M} . The channels Φ^X can now be chosen to be the same T_Y for all $X \in \Sigma$.

Example 8 (Predual instruments). Let \mathbf{M} , Y , \mathcal{M} , T , and Φ^X be as in Theorem 3 and Corollary 1. Let $\mathcal{M}_* : \Sigma \times \mathcal{T}(\mathcal{H}) \rightarrow \mathcal{T}(\mathcal{K})$, $T_* : \mathcal{T}(\mathcal{H}_\oplus) \rightarrow \mathcal{T}(\mathcal{K})$, and $\Phi_*^X : \mathcal{T}(\mathcal{H}) \rightarrow \mathcal{T}(\mathcal{K})$ be the predual mappings of \mathcal{M} , T , and Φ^X , respectively. Now, for all $X \in \Sigma$, $\rho \in \mathcal{T}(\mathcal{H})$, and $B \in \mathcal{L}(\mathcal{K})$,

$$\text{tr}[\mathcal{M}_*(X, \rho)B] = \text{tr}[\rho\mathcal{M}(X, B)] = \text{tr}[\rho Y^*\hat{\chi}_X T(B)\hat{\chi}_X Y] = \text{tr}[\hat{\chi}_X Y \rho Y^*\hat{\chi}_X T(B)]$$

so that

$$\mathcal{M}_*(X, \rho) = T_*(\hat{\chi}_X Y \rho Y^*\hat{\chi}_X),$$

that is, the Lüders operation $\rho \mapsto \hat{\chi}_X \tilde{\rho} \hat{\chi}_X$ of the larger system (with the Hilbert space \mathcal{H}_\oplus) first operates to a subsystem state $\tilde{\rho} = Y \rho Y^* \in \mathcal{S}(Y\mathcal{H})$, $\mathcal{H} \cong Y\mathcal{H} \subseteq \mathcal{H}_\oplus$ and, then, the resulting (nonnormalized) state is transformed by the channel T_* .

Similarly, we get

$$\mathcal{M}_*(X, \rho) = \Phi_*^X(\sqrt{\mathbf{M}(X)}\rho\sqrt{\mathbf{M}(X)})$$

which can be interpreted as a combination of a Lüders operation $\rho \mapsto \sqrt{\mathbf{M}(X)}\rho\sqrt{\mathbf{M}(X)}$ and a channel Φ_*^X . In the both above interpretations of \mathcal{M}_* , one can see the action of the channel as adding quantum noise to the Lüders operation.

5. MEASUREMENT THEORY

Following e.g. [31, Definition 3.1] or [4], we define a *measuring process* (a *measurement model* or a *premeasurement*) \mathbf{M} of a POVM $\mathbf{M} \in \text{Obs}(\Sigma, \mathcal{H})$ as a 4-tuple $\mathbf{M} = \langle \mathcal{H}', \mathbf{P}, \sigma, U \rangle$ consisting of a (possibly nonseparable) Hilbert space \mathcal{H}' attached to the probe, a PVM $\mathbf{P} : \Sigma \rightarrow \mathcal{L}(\mathcal{H}')$ (the pointer observable), an initial state $\sigma \in \mathcal{S}(\mathcal{H}')$ of the probe, and a unitary operator $U \in \mathcal{L}(\mathcal{H} \otimes \mathcal{H}')$ (the measurement interaction) satisfying the relation

$$\text{tr}[\rho\mathbf{M}(X)] = \text{tr}[U(\rho \otimes \sigma)U^*(I_{\mathcal{H}} \otimes \mathbf{P}(X))], \quad \rho \in \mathcal{T}(\mathcal{H}), \quad X \in \Sigma,$$

or, equivalently,

$$\mathbf{M}(X) = E_\sigma[U^*(I_{\mathcal{H}} \otimes \mathbf{P}(X))U], \quad X \in \Sigma,$$

where $E_\sigma : \mathcal{L}(\mathcal{H} \otimes \mathcal{K}) \rightarrow \mathcal{L}(\mathcal{H})$ is a (normal) CP map defined by the formula

$$\text{tr}[\rho E_\sigma(A)] := \text{tr}[(\rho \otimes \sigma)A], \quad \rho \in \mathcal{T}(\mathcal{H}), \quad A \in \mathcal{L}(\mathcal{H} \otimes \mathcal{K}).$$

Moreover, $\mathbf{M} = \langle \mathcal{H}', \mathbf{P}, \sigma, U \rangle$ is said to be *pure* (or *normal*) if $\sigma = |\xi\rangle\langle\xi|$ for some $\xi \in \mathcal{H}'$, $\|\xi\| = 1$.

A measuring process $\mathbf{M} = \langle \mathcal{H}', \mathbf{P}, \sigma, U \rangle$ of \mathbf{M} defines an \mathbf{M} -compatible instrument $\mathcal{M} \in \text{Ins}(\Sigma, \mathcal{H}, \mathcal{H})$ by

$$(3) \quad \mathcal{M}(X, B) := E_\sigma[U^*(B \otimes \mathbf{P}(X))U], \quad X \in \Sigma, \quad B \in \mathcal{L}(\mathcal{H}).$$

Now its predual instrument

$$\mathcal{M}_*(X, \rho) = \text{tr}_{\mathcal{H}'}[U(\rho \otimes \sigma)U^*(I_{\mathcal{H}} \otimes \mathbf{P}(X))], \quad X \in \Sigma, \quad \rho \in \mathcal{T}(\mathcal{H}),$$

so that $\text{tr}[\rho\mathbf{M}(X)] = \text{tr}[\mathcal{M}_*(X, \rho)]$. Two measuring processes are *statistically equivalent* if they define the same instrument.

A fundamental result of Ozawa [31, Theorem 5.1] is that for each instrument $\mathcal{M} \in \text{Ins}(\Sigma, \mathcal{H}, \mathcal{H})$ there exists a *pure* measuring process $\mathbf{M} = \langle \mathcal{H}', \mathbf{P}, |\xi\rangle\langle\xi|, U \rangle$ of the associate observable \mathbf{M} of

\mathcal{M} , that is, \mathcal{M} is of the form (3). Then we say that \mathbf{M} is a *pure realization of the instrument* \mathcal{M} . Moreover, we say that a pure realization $\mathbf{M} = \langle \mathcal{H}', \mathbf{P}, |\xi\rangle\langle\xi|, U \rangle$ of \mathcal{M} is *minimal* if for each pure realization $\overline{\mathbf{M}} = \langle \overline{\mathcal{H}}', \overline{\mathbf{P}}, |\overline{\xi}\rangle\langle\overline{\xi}|, \overline{U} \rangle$ of \mathcal{M} there exists an isometry $\mathcal{H}' \rightarrow \overline{\mathcal{H}}'$. Physically this means that the ancillary space \mathcal{H}' of a minimal \mathbf{M} is (up to a unitary equivalence) the smallest possible Hilbert space, i.e. there are no unnecessary degrees of freedom in the measuring process. Next we consider minimal pure realizations of instruments.

Example 9. Let $\mathcal{M} \in \text{Ins}(\Sigma, \mathcal{H}, \mathcal{H})$ with the isometry $Y : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}_\oplus$ of Theorem 2. Let \mathbf{h} be an ON basis of \mathcal{H} and fix a unit vector $\xi_\oplus \in \mathcal{H}_\oplus$. Since $\{Yh_n\}_{n=1}^{\dim \mathcal{H}}$ and $\{h_n \otimes \xi_\oplus\}_{n=1}^{\dim \mathcal{H}}$ are ON sets (with the same cardinality) one can define a unitary operator $U_{\xi_\oplus} \in \mathcal{L}(\mathcal{H} \otimes \mathcal{H}_\oplus)$ by setting $U_{\xi_\oplus}(h_n \otimes \xi_\oplus) := Yh_n$. Obviously, $U_{\xi_\oplus}(\psi \otimes \xi_\oplus) = Y\psi$ for all $\psi \in \mathcal{H}$ and we get, for all $\varphi, \psi \in \mathcal{H}$, $X \in \Sigma$, and $B \in \mathcal{L}(\mathcal{H})$,

$$\begin{aligned} \langle \varphi | E_{|\xi_\oplus\rangle\langle\xi_\oplus|} [U_{\xi_\oplus}^* (B \otimes \hat{\chi}_X) U_{\xi_\oplus}] \psi \rangle &= \text{tr} \left[U_{\xi_\oplus} (|\psi \otimes \xi_\oplus\rangle\langle\varphi \otimes \xi_\oplus|) U_{\xi_\oplus}^* (B \otimes \hat{\chi}_X) \right] \\ &= \langle \varphi | Y^* (B \otimes \hat{\chi}_X) Y \psi \rangle = \langle \varphi | \mathcal{M}(X, B) \psi \rangle \end{aligned}$$

so that $\mathbf{M}_{U_{\xi_\oplus}}^\mathcal{M} := \langle \mathcal{H}_\oplus, [X \mapsto \hat{\chi}_X], |\xi_\oplus\rangle\langle\xi_\oplus|, U_{\xi_\oplus} \rangle$ is a minimal pure realization of \mathcal{M} by the next Theorem 4.

Let $\mathcal{M} \in \text{Ins}(\Sigma, \mathcal{H}, \mathcal{H})$ with the (minimal) isometry $Y : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}_\oplus$ of Theorem 2, i.e. $\mathcal{M}(X, B) = Y^*(B \otimes \hat{\chi}_X)Y$ and the linear space

$$\mathcal{V} := \text{lin} \{ (B \otimes \hat{\chi}_X) Y \psi \mid B \in \mathcal{L}(\mathcal{H}), X \in \Sigma, \psi \in \mathcal{H} \}$$

is dense in $\mathcal{H} \otimes \mathcal{H}_\oplus$.

Theorem 4. *For any pure realization $\langle \mathcal{H}', \mathbf{P}, |\xi\rangle\langle\xi|, U \rangle$ of \mathcal{M} there exists a unique isometry $W : \mathcal{H}_\oplus \rightarrow \mathcal{H}'$ such that $\mathbf{P}(X)W = W\hat{\chi}_X$ for all $X \in \Sigma$ and $(I_{\mathcal{H}} \otimes W)Y\psi = U(\psi \otimes \xi)$ for all $\psi \in \mathcal{H}$. The ancillary space \mathcal{H}' is unitarily equivalent with the dilation space \mathcal{H}_\oplus if and only if W is unitary. Finally, $\dim \mathcal{H}' \geq \dim \mathcal{H}_\oplus \geq \dim \mathcal{H}$ and, if \mathcal{H}_\oplus is not separable, then \mathcal{H}' cannot be separable.*

Proof. Let $\langle \mathcal{H}', \mathbf{P}, |\xi\rangle\langle\xi|, U \rangle$ be a pure realization of \mathcal{M} . Since, for all $\varphi, \psi \in \mathcal{H}$, $X \in \Sigma$, and $B \in \mathcal{L}(\mathcal{H})$,

$$\langle \varphi | \mathcal{M}(X, B) \psi \rangle = \langle U(\varphi \otimes \xi) | (B \otimes \mathbf{P}(X)) U(\psi \otimes \xi) \rangle = \langle Y\varphi | (B \otimes \hat{\chi}_X) Y\psi \rangle$$

one can define a linear map

$$\overline{W} \left((B \otimes \hat{\chi}_X) Y \psi \right) := (B \otimes \mathbf{P}(X)) U(\psi \otimes \xi)$$

on \mathcal{V} which is well defined and extends to an isometry $\overline{W} : \mathcal{H} \otimes \mathcal{H}_\oplus \rightarrow \mathcal{H} \otimes \mathcal{H}'$ by the usual calculation:

$$\begin{aligned} \left\| \overline{W} \left(\sum_i (B_i \otimes \hat{\chi}_{X_i}) Y \psi_i \right) \right\|^2 &= \sum_{i,j} \langle U(\psi_i \otimes \xi) | (B_i^* B_j \otimes \mathbf{P}(X_i \cap X_j)) U(\psi_j \otimes \xi) \rangle \\ &= \sum_{i,j} \langle Y \psi_i | (B_i^* B_j \otimes \hat{\chi}_{X_i \cap X_j}) Y \psi_j \rangle = \left\| \sum_i (B_i \otimes \hat{\chi}_{X_i}) Y \psi_i \right\|^2. \end{aligned}$$

Since, for all $B \in \mathcal{L}(\mathcal{H})$, $X \in \Sigma$, and $(B' \otimes \hat{\chi}_{X'}) Y \psi \in \mathcal{V}$,

$$\begin{aligned} (B \otimes \mathbf{P}(X)) \overline{W} \left((B' \otimes \hat{\chi}_{X'}) Y \psi \right) &= (B \otimes \mathbf{P}(X)) (B' \otimes \mathbf{P}(X')) U(\psi \otimes \xi) \\ &= (BB' \otimes \mathbf{P}(X \cap X')) U(\psi \otimes \xi) = \overline{W} \left((B \otimes \hat{\chi}_X) (B' \otimes \hat{\chi}_{X'}) Y \psi \right) \end{aligned}$$

one gets $(B \otimes P(X))\overline{W} = \overline{W}(B \otimes \hat{\chi}_X)$ which implies (put $X = \Omega$) that $(B \otimes I_{\mathcal{H}'})\overline{W} = \overline{W}(B \otimes I_{\mathcal{H}_\oplus})$ and, hence, $\overline{W} = I_{\mathcal{H}} \otimes W$ where $W : \mathcal{H}_\oplus \rightarrow \mathcal{H}'$ is an isometry for which $P(X)W = W\hat{\chi}_X$ for all $X \in \Sigma$. Obviously, $(I_{\mathcal{H}} \otimes W)Y\psi = U(\psi \otimes \xi)$ follows from the definition of \overline{W} .

Suppose then that $W' : \mathcal{H}_\oplus \rightarrow \mathcal{H}'$ is such that $P(X)W' = W'\hat{\chi}_X$ for all $X \in \Sigma$ and $(I_{\mathcal{H}} \otimes W')Y\psi = U(\psi \otimes \xi)$ for all $\psi \in \mathcal{H}$. Then

$$(I_{\mathcal{H}} \otimes W')\left((B \otimes \hat{\chi}_X)Y\psi\right) = (B \otimes P(X))U(\psi \otimes \xi) = (I_{\mathcal{H}} \otimes W)\left((B \otimes \hat{\chi}_X)Y\psi\right)$$

so that $I_{\mathcal{H}} \otimes W' = I_{\mathcal{H}} \otimes W$ by the density of \mathcal{V} , and thus $W' = W$ showing the uniqueness of W .

Since $W^*W = I_{\mathcal{H}_\oplus}$, it follows that $R = WW^*$ is a projection on \mathcal{H}' , and $W\mathcal{H}_\oplus = R\mathcal{H}'$. Thus, $R\mathcal{H}'$ is unitarily equivalent with \mathcal{H}_\oplus , and if also \mathcal{H}' is unitarily equivalent with \mathcal{H}_\oplus it follows that $R\mathcal{H}'$ is unitarily equivalent with \mathcal{H}' implying $R = I_{\mathcal{H}'}$ and W is unitary. The remaining claims are trivial. \square

Immediately we get the following corollary:

Corollary 2. *Let $\mathbf{M} = \langle \mathcal{H}', P, |\xi\rangle\langle\xi|, U \rangle$ be a pure realization of \mathcal{M} . Then \mathbf{M} is minimal if and only if there exists a unitary operator $W : \mathcal{H}_\oplus \rightarrow \mathcal{H}'$ such that*

- $W^*P(X)W = \hat{\chi}_X$ for all $X \in \Sigma$,
- $(I_{\mathcal{H}} \otimes W)^*U(I_{\mathcal{H}} \otimes W) = U_{\xi_\oplus}$ where $\xi_\oplus = W^*\xi$ and $U_{\xi_\oplus} \in \mathcal{L}(\mathcal{H} \otimes \mathcal{H}_\oplus)$ is a unitary operator for which $U_{\xi_\oplus}(\psi \otimes \xi_\oplus) = Y\psi$ for all $\psi \in \mathcal{H}$.

Hence, a minimal pure realization of \mathcal{M} is always unitarily equivalent (and can be identified) with some $\mathbf{M}_{U_{\xi_\oplus}}^M = \langle \mathcal{H}_\oplus, [X \mapsto \hat{\chi}_X], |\xi_\oplus\rangle\langle\xi_\oplus|, U_{\xi_\oplus} \rangle$.

Remark 6. In this remark, we determine the structure of pure realizations which are not necessarily minimal.

Let $\mathbf{M} = \langle \mathcal{H}', P, |\xi\rangle\langle\xi|, U \rangle$ be a pure realization of \mathcal{M} with the isometry W of Theorem 4. Let $R = WW^*$ be the projection $R : \mathcal{H}' \rightarrow R\mathcal{H}' \cong \mathcal{H}_\oplus$, $R^\perp := I_{\mathcal{H}'} - R$, and $Y' := (I_{\mathcal{H}} \otimes W)Y$. It is easy to verify that, for all $X \in \Sigma$ and $B \in \mathcal{L}(\mathcal{H})$,

- (1) $W^*P(X)W = \hat{\chi}_X$,
- (2) $P(X)R = RP(X) = RP(X)R = W\hat{\chi}_XW^*$,
- (3) $P(X) = RP(X)R + R^\perp P(X)R^\perp$,
- (4) $E_{|\xi\rangle\langle\xi|}[U^*(B \otimes P(X))U] = Y'^*(B \otimes P(X))Y' = Y^*(B \otimes \hat{\chi}_X)Y = \mathcal{M}(X, B)$,
- (5) $(I_{\mathcal{H}} \otimes R)Y' = Y'$,
- (6) $\mathcal{M}(X, B) = Y'^*(B \otimes RP(X)R)Y'$,
- (7) $(I_{\mathcal{H}} \otimes R)U(\psi \otimes \xi) = U(\psi \otimes \xi)$, $\psi \in \mathcal{H}$.

We have two cases a) $\xi \in R\mathcal{H}'$ and b) $\xi \notin R\mathcal{H}'$. In the a) case one sees that, if the compound system is in a (pure) state $\psi \otimes \xi \in \mathcal{H} \otimes R\mathcal{H}'$ before the measurement, then the measurement coupling U transforms it to the state $U(\psi \otimes \xi) \in \mathcal{H} \otimes R\mathcal{H}'$ by item (7) above. Now, instead of P , one can equally well use the projected pointer observable (PVM) $\Sigma \ni X \mapsto RP(X)R \in \mathcal{L}(R\mathcal{H}')$ of the smaller probe system, see (6); the PVM $X \mapsto R^\perp P(X)R^\perp$ is irrelevant in the measurement model \mathbf{M} , see (3).

In the case b), one can slightly modify U in the following way: Let $\tilde{\xi} \in R\mathcal{H}$ be a unit vector and $W_0 \in \mathcal{L}(\mathcal{H}')$ a unitary rotation operator which maps ξ to $\tilde{\xi}$ ($= W_0\xi$). Define a unitary operator $\tilde{U} := U(I_{\mathcal{H}} \otimes W_0^*)$ for which $\tilde{U}(\psi \otimes \tilde{\xi}) = U(\psi \otimes \xi)$ for all $\psi \in \mathcal{H}$. From (7) one sees that $(I_{\mathcal{H}} \otimes R)\tilde{U}(\psi \otimes \tilde{\xi}) = \tilde{U}(\psi \otimes \tilde{\xi})$, $\psi \in \mathcal{H}$. Hence, we get a measuring process $\langle \mathcal{H}', P, |\tilde{\xi}\rangle\langle\tilde{\xi}|, \tilde{U} \rangle$ of type a), $\tilde{\xi} \in R\mathcal{H}'$, which is statistically equivalent with \mathbf{M} .

Example 10 (The standard model). In some physically relevant cases, the measurement coupling U is given and then one determines observables which can be measured by using U and varying pointer observables and initial probe states. For example, in the (generalized) standard model of quantum measurement theory U is of the form $U = e^{i\lambda(A \otimes B)}$ (i.e. U is the standard measurement coupling) where A is a selfadjoint operator of the system on which a measurement is performed, B is a selfadjoint operator of the ancillary space, and $\lambda \in \mathbb{R}$ is a coupling constant (see, e.g. [3, 15]). It can be shown [3] that, when the pointer observable is a selfadjoint operator then one usually measures a smeared (or unsharp) version of A . An important example is a standard measurement of a (fuzzy) position observable. In this example, we generalize the standard measurement model for arbitrary POVMs.

From Remark 6 and Corollary 2, one sees that we can study a pure measurement model $\mathbf{M} = \langle \mathcal{H}_\oplus, [X \mapsto \hat{\chi}_X], |\xi\rangle\langle\xi|, U \rangle$ where \mathcal{H}_\oplus is an arbitrary direct integral Hilbert space and U is the standard measurement coupling, $U = e^{i\lambda(A \otimes B)}$. Now A and B are selfadjoint operators of \mathcal{H} and \mathcal{H}_\oplus , respectively. Define $Y\psi := U(\psi \otimes \xi)$ for all $\psi \in \mathcal{H}$. If $\mathbf{M}_A \in \text{Obs}(\mathcal{B}(\mathbb{R}), \mathcal{H})$ is a spectral measure of A we get, for all $\psi \in \mathcal{H}$,

$$Y\psi = \int_{\mathbb{R}} \mathbf{M}_A(da) \psi \otimes \xi(a)$$

where $\xi(a) := e^{ia\lambda B} \xi$ is a unit vector for each $a \in \mathbb{R}$. Now the realized instrument $\mathcal{M}(X, B) = Y^*(B \otimes \hat{\chi}_X)Y$, $X \in \Sigma$, $B \in \mathcal{L}(\mathcal{H})$, or

$$\begin{aligned} \langle \psi | \mathcal{M}(X, B) \psi \rangle &= \langle Y\psi | (B \otimes \hat{\chi}_X) Y\psi \rangle = \int_{\mathbb{R}} \int_{\mathbb{R}} \langle \mathbf{M}_A(da) \psi \otimes \xi(a) | (B \otimes \hat{\chi}_X) \mathbf{M}_A(da') \psi \otimes \xi(a') \rangle \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \langle \xi(a) | \hat{\chi}_X \xi(a') \rangle \langle \psi | \mathbf{M}_A(da) B \mathbf{M}_A(da') \psi \rangle. \end{aligned}$$

Especially, the measured observable $\mathbf{M} \in \text{Obs}(\Sigma, \mathcal{H})$ is given by

$$\langle \psi | \mathbf{M}(X) \psi \rangle = \langle \psi | \mathcal{M}(X, I_{\mathcal{H}}) \psi \rangle = \int_{\mathbb{R}} \|\hat{\chi}_X \xi(a)\|^2 \langle \psi | \mathbf{M}_A(da) \psi \rangle.$$

that is,

$$\mathbf{M}(X) = \int_{\mathbb{R}} K(X, a) \mathbf{M}_A(da)$$

where $K : \Sigma \times \mathbb{R} \rightarrow [0, 1]$,

$$K(X, a) := \|\hat{\chi}_X \xi(a)\|^2 = \|\hat{\chi}_X e^{ia\lambda B} \xi\|^2 = \int_X k(x, a) d\mu(x),$$

is a *Markov kernel* (or a conditional or transition probability) with the density $k(x, a) := \|(e^{ia\lambda B} \xi)(x)\|^2$: for all $a \in \mathbb{R}$,

$$\Sigma \ni X \mapsto K(X, a) \in [0, 1]$$

is a probability measure and, for all $X \in \Sigma$,

$$\mathbb{R} \ni a \mapsto K(X, a) \in [0, 1]$$

is continuous. If B (its spectral measure) commutes with the canonical spectral measure $X \mapsto \hat{\chi}_X$ then $K(X, a) = \|\hat{\chi}_X \xi\|^2$ does not depend on a and the measured POVM $\mathbf{M}(X) = \|\hat{\chi}_X \xi\|^2 I_{\mathcal{H}}$ is trivial. Hence, to measure some nontrivial POVM, the probe observables B and $X \mapsto \hat{\chi}_X$ must be ‘complementary’ in some sense (e.g. position and momentum [3]).

Finally we note that, on the first hand, since (as a PVM) \mathbf{M}_A is commutative, \mathbf{M} is also commutative. On the other hand, some important POVMs (e.g. the canonical phase observable) are (even totally) noncommutative. We may conclude that the standard measurement couplings cannot be used in any measurement of noncommutative observables.

5.1. Posterior states. Let $\mathcal{M} \in \text{Ins}(\Sigma, \mathcal{K}, \mathcal{H})$ with the (minimal) point- and setwise Kraus operators of Theorem 2, i.e., we may write either

$$\langle \varphi | \mathcal{M}(X, B) \psi \rangle = \int_X \sum_{k=1}^{n(x)} \langle A_k(x) \varphi | B A_k(x) \psi \rangle d\mu(x), \quad \varphi, \psi \in V_{\mathbf{h}},$$

or

$$\mathcal{M}(X, B) = \sum_{k=1}^{n(X)} A_k(X)^* B A_k(X) \quad (\text{ultraweakly})$$

where, for all $x \in \Omega$, the operators $A_k(x) : V_{\mathbf{h}} \rightarrow \mathcal{K}$ are linearly independent, and for all $X \in \Sigma$, the bounded operators $A_k(X) : \mathcal{H} \rightarrow \mathcal{K}$ are linearly independent. Note that, usually in quantum measurement theory, one assumes $\mathcal{K} = \mathcal{H}$.

Let $\rho \in \mathcal{S}(\mathcal{H})$ be a (fixed) ‘initial state’ of the system. Then the ‘measurement’ probability measure is

$$X \mapsto \mu_{\rho}^{\mathcal{M}}(X) := \text{tr}[\rho \mathcal{M}_{I_{\mathcal{K}}}(X)] = \text{tr}[\rho \mathcal{M}(X, I_{\mathcal{K}})] = \text{tr}[\mathcal{M}_*(X, \rho)].$$

Denote by w_{ρ} the (nonnegative) Radon-Nikodým derivative of $\mu_{\rho}^{\mathcal{M}}$ with respect to μ , i.e. $d\mu_{\rho}^{\mathcal{M}}(x) = w_{\rho}(x)d\mu(x)$. Following [32], we define a *conditional expectation* $\{E(B|x) | x \in \Omega\}$ of $B \in \mathcal{L}(\mathcal{K})$ with respect to (\mathcal{M}, ρ) as a (complex-valued) measurable function $x \mapsto E(B|x)$ such that, for all $X \in \Sigma$,

$$\int_X E(B|x) d\mu_{\rho}^{\mathcal{M}}(x) = \text{tr}[\rho \mathcal{M}(X, B)] = \text{tr}[B \mathcal{M}_*(X, \rho)],$$

and we call $\{\rho_x | x \in \Omega\} \subseteq \mathcal{S}(\mathcal{K})$ a family of *posterior states with respect to* (\mathcal{M}, ρ) if, for all $B \in \mathcal{L}(\mathcal{H})$, one has $\text{tr}[\rho_x B] = E(B|x)$ for $\mu_{\rho}^{\mathcal{M}}$ -almost all $x \in \Omega$. If $\{\rho'_x | x \in \Omega\} \subseteq \mathcal{S}(\mathcal{K})$ is another family of posterior states then $\rho'_x = \rho_x$ for $\mu_{\rho}^{\mathcal{M}}$ -almost all $x \in \Omega$.

Let $\mathbf{h} = \{h_n\}$ be an ON basis of \mathcal{H} such that it diagonalizes ρ , that is,

$$\rho = \sum_{n=1}^{\dim \mathcal{H}} \lambda_n |h_n\rangle \langle h_n|, \quad \lambda_n \geq 0, \quad \sum_{n=1}^{\dim \mathcal{H}} \lambda_n = 1.$$

Now, for each positive $B \in \mathcal{L}(\mathcal{K})$,

$$\begin{aligned} \int_X \text{tr}[\rho_x B] w_{\rho}(x) d\mu(x) &= \text{tr}[\rho \mathcal{M}(X, B)] = \sum_{n=1}^{\dim \mathcal{H}} \lambda_n \langle h_n | \mathcal{M}(X, B) h_n \rangle \\ &= \sum_{n=1}^{\dim \mathcal{H}} \lambda_n \int_X \sum_{k=1}^{n(x)} \langle A_k(x) h_n | B A_k(x) h_n \rangle d\mu(x) \end{aligned}$$

for all $X \in \Sigma$, so that

$$\text{tr}[\rho_x B] w_{\rho}(x) = \sum_{n=1}^{\dim \mathcal{H}} \lambda_n \sum_{k=1}^{n(x)} \langle A_k(x) h_n | B A_k(x) h_n \rangle$$

for μ -almost all $x \in \Omega$ by the monotonic convergence theorem. For $B = I_{\mathcal{K}}$, one gets

$$w_{\rho}(x) = \sum_{n=1}^{\dim \mathcal{H}} \lambda_n \sum_{k=1}^{n(x)} \|A_k(x) h_n\|^2$$

for μ -almost all $x \in \Omega$. Since any $B \in \mathcal{L}(\mathcal{H})$ can be decomposed into positive parts, one sees that, if $w_{\rho}(x) \neq 0$, then the posterior state corresponding to the ‘outcome’ $x \in \Omega$ can be chosen

to be

$$\rho_x = w_\rho(x)^{-1} \sum_{n=1}^{\dim \mathcal{H}} \lambda_n \sum_{k=1}^{n(x)} |\mathbf{A}_k(x)h_n\rangle \langle \mathbf{A}_k(x)h_n| = w_\rho(x)^{-1} \sum_{k=1}^{n(x)} \mathbf{A}_k(x)\rho\mathbf{A}_k(x)^* \quad (\text{formally}).$$

This is a generalization of Holevo's result [20]. Sometimes (in the case $\mathcal{K} = \mathcal{H}$) ρ_x is interpreted as a state conditioned upon an outcome x of the measurement described by \mathcal{M} , that is, this measurement of the POVM $\mathbf{M}_{I_{\mathcal{H}}}$ changes the initial state ρ into a posterior state ρ_x if the value x is observed [32, 20]. This interpretation is problematic since, on the first hand, ρ_x is not necessarily unique. On the other hand, even if $\{x\} \in \Sigma$ it may happen that $\mu(\{x\}) = 0$ (e.g. position observables). Then it is better to define a *conditional output state*

$$\rho_X = \mathcal{M}_*(X, \rho) / \text{tr}[\mathcal{M}_*(X, \rho)] = \mu_\rho^{\mathcal{M}}(X)^{-1} \sum_{k=1}^{n(X)} \mathbf{A}_k(X)\rho\mathbf{A}_k(X)^* \in \mathcal{S}(\mathcal{K})$$

corresponding to a set $X \in \Sigma$ of outcomes (if $\text{tr}[\mathcal{M}_*(X, \rho)] > 0$), which describes the state, at the instant after the measurement, of the subensemble of the measured system in which the outcomes of the measurement lie in X . However, even if $X \subset X'$ it often happens that $\rho_X \neq \rho_{X'}$.

Notice that

$$\rho_X = \mu_\rho^{\mathcal{M}}(X)^{-1} \int_X \rho_x d\mu_\rho^{\mathcal{M}}(x)$$

so that $\rho_{\{x\}} = \rho_x$ if and only if $\{x\} \in \Sigma$ and $\mu_\rho^{\mathcal{M}}(\{x\}) > 0$. Hence, in the discrete case, there is no above mentioned problems. The most familiar example is of course the (strongly repeatable) von Neumann-Lüders instrument

$$\mathcal{M}(X, B) = \sum_{i \in X} P_i B P_i$$

where $\{P_i\}$ constitute a discrete PVM (see, e.g. [11, 10, 31, 4]). Now if the value i is observed then the state at the instant after the measurement is

$$\rho_i = \rho_{\{i\}} = \frac{P_i \rho P_i}{\text{tr}[\rho P_i]}.$$

Finally, we recall that the von Neumann-Lüders instrument is extreme [9].

6. REFINEMENTS AND RANK 1 INSTRUMENTS

Let $\mathcal{M} \in \text{Ins}(\Sigma, \mathcal{K}, \mathcal{H})$, and $\mathcal{H}_\oplus = \int_\Omega^\oplus \mathcal{H}_{n(x)} d\mu(x)$ the direct integral and operators $\mathbf{A}_k(x)$ the pointwise Kraus operators of Theorem 2 associated to \mathcal{M} . Let

$$\mathbf{N} := \{k \in \mathbb{N}_\infty \mid 1 \leq k \leq \dim \mathcal{H} \dim \mathcal{K}\}$$

and $\# : 2^{\mathbf{N}} \rightarrow \mathbb{N}_\infty$ be the counting measure, i.e. $\#N$ is the number of the elements of $N \subseteq \mathbf{N}$. Let $\# \times \mu$ be the product measure defined on the product σ -algebra $\Sigma_{\mathbf{N}} \subseteq 2^{\mathbf{N} \times \Omega}$ of $2^{\mathbf{N}}$ and Σ . Define $\mathbf{A}(k, x) := \mathbf{A}_k(x)$ when $k \leq n(x)$ and $\mathbf{A}(k, x) := 0$ for $k > n(x)$. Then, for all $\varphi, \psi \in V_{\mathbf{h}}$,

$$\begin{aligned} \langle \varphi | \mathcal{M}(X, B) \psi \rangle &= \int_X \sum_{k=1}^{n(x)} \langle \varphi | \mathbf{A}_k(x)^* B \mathbf{A}_k(x) \psi \rangle d\mu(x) \\ &= \int_{\mathbf{N} \times X} \langle \varphi | \mathbf{A}(k, x)^* B \mathbf{A}(k, x) \psi \rangle d(\# \times \mu)(k, x) \end{aligned}$$

and we can define the *maximally refined* (rank-1) instrument $\mathcal{M}_1 \in \text{Ins}(\Sigma_{\mathbf{N}}, \mathcal{K}, \mathcal{H})$ of \mathcal{M} by

$$\langle \varphi | \mathcal{M}_1(\overline{X}, B) \psi \rangle := \int_{\overline{X}} \langle \varphi | \mathbf{A}(k, x)^* B \mathbf{A}(k, x) \psi \rangle d(\# \times \mu)(k, x), \quad \varphi, \psi \in V_{\mathbf{h}}, \quad \overline{X} \in \Sigma_{\mathbf{N}}, \quad B \in \mathcal{L}(\mathcal{K}).$$

Proposition 3. *If $\mathcal{M} \in \text{Ext Ins}(\Sigma, \mathcal{K}, \mathcal{H})$ then $\mathcal{M}_1 \in \text{Ext Ins}(\Sigma_{\mathbf{N}}, \mathcal{K}, \mathcal{H})$.*

Proof. Let $\mathcal{M} \in \text{Ins}(\Sigma, \mathcal{K}, \mathcal{H})$ be extreme. From Remark 2 one sees that, for any decomposable operator $D \in \mathcal{L}(\mathcal{H}_{\oplus})$, the condition

$$\int_{\Omega} \sum_{k,l=1}^{n(x)} D(x)_{kl} \langle \varphi | \mathbf{A}_k(x)^* \mathbf{A}_l(x) | \psi \rangle d\mu(x) = 0, \quad \varphi, \psi \in V_{\mathbf{h}},$$

implies $D = 0$. Especially, by choosing all operators $D(x)$ diagonal, i.e., $D(x)_{kl} = \delta_{kl} d(k, x)$ for all $x \in \Omega$ (where $d \in L^{\infty}(\# \times \mu)$ is such that $\|D\| = \text{ess sup}\{|d(k, x)| \mid x \in \Omega, k < n(x)+1\} < \infty$) we get that the condition

$$\int_{\mathbf{N} \times \Omega} d(k, x) \langle \varphi | \mathbf{A}(k, x)^* \mathbf{A}(k, x) \psi \rangle d(\# \times \mu)(k, x) = \int_{\Omega} \sum_{k,l=1}^{n(x)} \delta_{kl} f_k(x) \langle \varphi | \mathbf{A}_k(x)^* \mathbf{A}_l(x) | \psi \rangle d\mu(x) = 0$$

implies $d = 0$. Hence, $\mathcal{M}_1 \in \text{Ext Ins}(\Sigma_{\mathbf{N}}, \mathcal{K}, \mathcal{H})$ by Remark 2. \square

As a special case ($\mathcal{K} = \mathbb{C}$), one sees that any POVM can be maximally refined (recall also Example 1). Thus, next we consider the instruments and measuring processes of rank-1 POVMs. Note that important examples of rank-1 POVMs are position and momentum observables (of a spin 0 particle moving on a space manifold), rotated quadratures, phase space observables generated by pure states, the canonical phase observable of a single mode electromagnetic field, and many important discrete observables. Often their related instruments are also of rank-1 (see, e.g. Section 4.6 of [10]).

Let $\mathbf{M} \in \text{Obs}(\Sigma, \mathcal{H})$ be a rank-1 POVM, that is, in Theorem 1, $n(x) \in \{0, 1\}$ for all $x \in \Omega$, $\mathcal{H}_1 \cong \mathbb{C}$, $\mathcal{H}_0 = \{0\}$, $\mathcal{H}_{\oplus} = L^2(\mu)$, and for all $X \in \Sigma$ we have

$$\mathbf{M}(X) = \sum_{n,m=1}^{\dim \mathcal{H}} \int_X \overline{\psi_n(x)} \psi_m(x) d\mu(x) |h_n\rangle \langle h_m|$$

or

$$\langle \varphi | \mathbf{M}(X) \psi \rangle = \int_X \langle \varphi | d_1(x) \rangle \langle d_1(x) | \psi \rangle d\mu(x), \quad \varphi, \psi \in V_{\mathbf{h}}.$$

Let $\mathcal{M} \in \text{Ins}(\Sigma, \mathcal{K}, \mathcal{H})$. Then, by Theorem 3, $\mathcal{M}(X, I_{\mathcal{K}}) \equiv \mathbf{M}(X)$ if and only if there exists a decomposable CP channel $T : \mathcal{L}(\mathcal{K}) \rightarrow \mathcal{L}(L^2(\mu))$, $B \mapsto T(B) = \int_{\Omega}^{\oplus} \text{tr}[\sigma_x B] d\mu(x)$, where $\sigma_x \in \mathcal{S}(\mathcal{K})$ are states for μ -almost all $x \in \Omega$, such that

$$\mathcal{M}(X, B) \equiv \sum_{n,m=1}^{\dim \mathcal{H}} \int_X \text{tr}[\sigma_x B] \overline{\psi_n(x)} \psi_m(x) d\mu(x) |h_n\rangle \langle h_m| = \int_X \text{tr}[\sigma_x B] d\mathbf{M}(x).$$

Any channel $B \mapsto \text{tr}[\sigma_x B]$ above can be chosen to be of the form (see, Theorem 3)

$$\text{tr}[\sigma_x B] = \langle \eta_x | (B \otimes I_{\mathcal{H}'_{n'(x)}}) \eta_x \rangle, \quad B \in \mathcal{L}(\mathcal{K}),$$

where $\eta_x \in \mathcal{K} \otimes \mathcal{H}'_{n'(x)}$ is a unit vector and $\mathcal{H}'_{\oplus} = \int_{\Omega}^{\oplus} \mathcal{H}'_{n'(x)} d\mu(x)$ is the direct integral Hilbert space of Theorem 2 associated to \mathcal{M} . Hence, we have shown that *any compatible instrument*

of a rank-1 POVM is a nuclear instrument (see, Example 7) and thus can be identified with (the equivalence class of) $\{\sigma_x\}_{x \in \Omega}$. By writing (for μ -almost all $x \in \Omega$)

$$\eta_x = \sum_{k=1}^{n'(x)} \varphi_k(x) \otimes b_k$$

we see that

$$\text{tr}[\sigma_x B] = \sum_{k=1}^{n'(x)} \langle \varphi_k(x) | B \varphi_k(x) \rangle, \quad \sigma_x = \sum_{k=1}^{n'(x)} |\varphi_k(x)\rangle \langle \varphi_k(x)|$$

and $n'(x)$ is the rank of the state σ_x . Hence, \mathcal{M} is of rank-1 (i.e. $n'(x) \in \{0, 1\}$) if and only if σ_x is a pure state for μ -almost everywhere. In this case one may choose $\mathcal{H}'_{\oplus} = \mathcal{H}_{\oplus}$.

One can also use another decomposition

$$\eta_x = \sum_{s=1}^{\dim \mathcal{K}} k_s \otimes \eta^s(x), \quad \sum_{s=1}^{\dim \mathcal{K}} \|\eta^s(x)\|^2 = 1,$$

to get

$$\text{tr}[\sigma_x B] = \sum_{s,t=1}^{\dim \mathcal{K}} B_{st} \langle \eta^s(x) | \eta^t(x) \rangle, \quad \sigma_x = \sum_{s,t=1}^{\dim \mathcal{K}} \langle \eta^s(x) | \eta^t(x) \rangle |k_t\rangle \langle k_s|,$$

and

$$\mathcal{M}(X, B) \equiv \sum_{n,m=1}^{\dim \mathcal{H}} \sum_{s,t=1}^{\dim \mathcal{K}} \int_X B_{st} \langle \psi_n(x) \eta^s(x) | \psi_m(x) \eta^t(x) \rangle d\mu(x) |h_n\rangle \langle h_m|$$

so that $\psi_n^s(x) = \psi_n(x) \eta^s(x) \in \mathcal{H}'_{n'(x)}$, $\sum_s k_s \otimes \psi_n^s(x) = \psi_n(x) \eta_x$, and the generalized vectors of \mathcal{M} can be chosen to be

$$d_k^s(x) = c_{1,sk}^x d_1(x), \quad c_{1,sk}^x = \langle \eta^s(x) | b_k \rangle, \quad \sum_{s,k} |c_{1,sk}^x| = 1.$$

If \mathcal{M} is also of rank 1 then $\eta_x \in \mathcal{K} \cong \mathcal{K} \otimes \mathcal{H}'_1$, $\eta^s(x) \in \mathbb{C}$, and $\sigma_x = |\eta_x\rangle \langle \eta_x|$ for μ -almost everywhere. Note that, following Holevo [21], one sees that *any EB channel can be seen as a rank-1 nuclear instrument whose associate observable is of rank-1*.

Theorem 5. *Let $\mathbf{M} \in \text{Obs}(\Sigma, \mathcal{H})$ be a rank-1 POVM and $\mathcal{M} \in \text{Ins}(\Sigma, \mathcal{K}, \mathcal{H})$ any \mathbf{M} -compatible instrument. Then \mathcal{M} is nuclear. If $\mathcal{M} \in \text{Ext Ins}(\Sigma, \mathcal{K}, \mathcal{H})$ then $\mathbf{M} \in \text{Ext Obs}(\Sigma, \mathcal{H})$. If \mathcal{M} is also of rank 1, then $\mathbf{M} \in \text{Ext Obs}(\Sigma, \mathcal{H})$ implies $\mathcal{M} \in \text{Ext Ins}(\Sigma, \mathcal{K}, \mathcal{H})$.*

Proof. Since $\mathcal{M} \in \text{Ext Ins}(\Sigma, \mathcal{K}, \mathcal{H})$ if and only if the condition

$$(4) \quad \sum_{s=1}^{\dim \mathcal{K}} \int_{\Omega} \langle \psi_n^s(x) | D(x) \psi_m^s(x) \rangle d\mu(x) = \int_{\Omega} \sum_{s=1}^{\dim \mathcal{K}} \langle \eta^s(x) | D(x) \eta^s(x) \rangle \overline{\psi_n(x)} \psi_m(x) d\mu(x) = 0$$

for all n, m , implies $D = 0$ (see, Remark 2). Let $d \in L^\infty(\mu)$ and set $D(x) = d(x) I_{\mathcal{H}'_{n'(x)}}$ to get that

$$\int_{\Omega} d(x) d\mathbf{M}(x) = 0$$

implies $d = 0$, i.e. \mathbf{M} is extreme.

From (4), one sees that if \mathbf{M} and (an \mathbf{M} -compatible) \mathcal{M} are of rank 1 then their extremality conditions are exactly the same condition: $\int_{\Omega} d(x) d\mathbf{M}(x) = 0$ (where $d \in L^\infty(\mu)$) implies $d = 0$. \square

7. MODULES OVER C^* -ALGEBRAS AND EXTREME KERNELS AND CP-MAPS

In this section, we follow [34]. We let A and A_e be C^* -algebras and assume that A_e is unital with the unit $e \in A_e$.

If M is a (right) Hilbert C^* -module over A and $B : M \rightarrow M$ a bounded A -linear⁴ map then B is a positive element of the C^* -algebra $L(M)$ of adjointable maps on M (i.e. $B \geq 0$) if and only if $\langle v|Bv \rangle \geq 0$ for all $v \in M$ if and only if $B = C^*C$ where $C : M \rightarrow M'$ is adjointable (and M' a Hilbert C^* -module over A) [30, Proposition 2.1.3]. We denote $B \leq B'$ if $B' - B \geq 0$, and let I_M be the identity (operator) of $L(M)$.

Let $n \in \mathbb{N}_+$ and $M_n(A)$ be the matrix C^* -algebra consisting of the A -valued $n \times n$ -matrices $(a_{ij})_{i,j=1}^n$. An element $(a_{ij})_{i,j=1}^n$ of $M_n(A)$ is positive if and only if $\sum_{i,j=1}^n a_i^* a_{ij} a_j \geq 0$ for all $a_1, \dots, a_n \in A$.

Let V be an A -module (i.e. a right module over the algebra A) and $S_A(V)$ the \mathbb{C} -linear space of A -sesquilinear maps $s : V \times V \rightarrow A$. Let $M_n(S_A(V))$ be the \mathbb{C} -linear space of $n \times n$ -matrices $(s_{ij})_{i,j=1}^n$ where the matrix elements s_{ij} belong to $S_A(V)$. Note that the matrix multiplication is not defined. We say that $(s_{ij})_{i,j=1}^n \in M_n(S_A(V))$ is positive if the matrix $(s_{ij}(v_i, v_j))_{i,j=1}^n$ is a positive element of $M_n(A)$ for all $v_1, \dots, v_n \in V$.

7.1. Positive-definite kernels. For any set $X \neq \emptyset$, we say that a mapping $K : X \times X \rightarrow S_A(V)$ is a *positive-(semi)definite A -kernel* if, for all $n \in \mathbb{N}_+$ and $x_1, \dots, x_n \in X$, the matrix $(K(x_i, x_j))_{i,j=1}^n \in M_n(S_A(V))$ is positive, i.e. for all $v_1, \dots, v_n \in V$

$$\sum_{i,j=1}^n [K(x_i, x_j)](v_i, v_j) \geq 0.$$

Let $\mathbf{K}(X, V)$ denote the convex set of all positive-definite A -kernels $K : X \times X \rightarrow S_A(V)$. The following theorem is proved in [34]:

Theorem 6. *For each $K \in \mathbf{K}(X, V)$ there exists a Hilbert C^* -module M over A and A -linear maps $D(x) : V \rightarrow M$, $x \in X$, such that*

- (i) $[K(x, x')](v, v') = \langle D(x)v | D(x')v' \rangle$, $x, x' \in X$, $v, v' \in V$,
- (ii) $\text{lin}_{\mathbb{C}} \cup_{x \in X} D(x)V$ is dense in M .

We say that (M, D) is a minimal Kolmogorov decomposition (MKD) for K .

If (M', D') is another MKD for K then there exists a unitary $U : M \rightarrow M'$ such that $UD(x) = D'(x)$ for all $x \in X$.

Any $K \in \mathbf{K}(X, V)$ is said to be *regular* if the Hilbert C^* -module M (associated with a MKD (M, D) of K) is self-dual. Note that *any* $K \in \mathbf{K}(X, V)$ is regular if A is finite dimensional (e.g. when $A \subseteq \mathcal{L}(\mathcal{H})$ where \mathcal{H} is a finite dimensional Hilbert space) [30, Section 2.5].

Fix $K_1 \in \mathbf{K}(X, V)$ and $Z \subseteq X \times X$. Let $\mathbf{C}(K_1, Z) \subseteq \mathbf{K}(X, V)$ consist of positive-definite A -kernels $K : X \times X \rightarrow S_A(V)$ such that $K(x, x') = K_1(x, x')$ for all $(x, x') \in Z$. Obviously, $\mathbf{C}(K_1, Z)$ is convex and we denote by $\text{Ext } \mathbf{C}(K_1, Z)$ its extreme points. Note that $\mathbf{C}(K_1, \emptyset) = \mathbf{K}(X, V)$.

Theorem 7. *Let $K \in \mathbf{C}(K_1, Z)$ be regular and (M, D) its MKD. Then $K \in \text{Ext } \mathbf{C}(K_1, Z)$ if and only if, for all selfadjoint $B \in L(M)$, the condition*

$$\langle D(x)v | BD(x')v' \rangle = 0, \quad (x, x') \in Z, \quad v, v' \in V,$$

implies $B = 0$.

⁴We assume implicitly that A -(sesqui)linear maps are also \mathbb{C} -(sesqui)linear.

Proof. For any $J, K \in \mathbf{K}(X, V)$ we denote $J \leq K$ if $K - J \in \mathbf{K}(X, V)$. Let $K \in \mathbf{C}(K_1, Z)$ be regular and (M, D) its MKD. Suppose that

$$K = \frac{1}{2}K_+ + \frac{1}{2}K_-$$

where $K_{\pm} \in \mathbf{C}(K_1, Z)$ and $K_+ \neq K_-$. Then $K_{\pm} \leq 2K$. Let (M_{\pm}, D_{\pm}) be a MKD of K_{\pm} so that, for all $n \in \mathbb{N}_+$, $x_1, \dots, x_n \in X$, and $v_1, \dots, v_n \in V$, $\sum_{i,j=1}^n [K_{\pm}(x_i, x_j)](v_i, v_j) \leq 2 \sum_{i,j=1}^n [K(x_i, x_j)](v_i, v_j)$ or, equivalently,

$$\left\langle \sum_{i=1}^n D_{\pm}(x_i)v_i \middle| \sum_{j=1}^n D_{\pm}(x_j)v_j \right\rangle \leq 2 \left\langle \sum_{i=1}^n D(x_i)v_i \middle| \sum_{j=1}^n D(x_j)v_j \right\rangle.$$

Hence, we may define \mathbb{C} -linear maps $G_{\pm} : M \rightarrow M_{\pm}$ by

$$G_{\pm}(D(x)v) := D_{\pm}(x)v, \quad x \in X, v \in V,$$

for which $\langle G_{\pm}w | G_{\pm}w \rangle \leq 2\langle w | w \rangle$ for all $w \in \text{lin}_{\mathbb{C}} \cup_{x \in X} D(x)V \subseteq M$. It follows that G_{\pm} is well defined, A -linear, and bounded (with the norm $\|G_{\pm}\| \leq \sqrt{2}$) [30, Theorem 2.1.4, Corollary 2.1.6]. By regularity of K the module M is self-dual and thus G_{\pm} is adjointable with the adjoint $G_{\pm}^* : M_{\pm} \rightarrow M$ [30, Proposition 2.5.2]. Define a positive $B_{\pm} := G_{\pm}^* G_{\pm} \in L(M)$ so that

$$[K_{\pm}(x, x')](v, v') = \langle G_{\pm}D(x)v | G_{\pm}D(x')v' \rangle = \langle D(x)v | G_{\pm}^* G_{\pm} D(x')v' \rangle = \langle D(x)v | B_{\pm} D(x')v' \rangle$$

for all $x, x' \in X$ and $v, v' \in V$. Let $B := B_+ - B_- \in L(M)$ for which $B^* = B$. Since $K_{\pm} \in \mathbf{C}(K_1, Z)$ one has

$$\begin{aligned} \langle D(x)v | BD(x')v' \rangle &= \langle D(x)v | B_+ D(x')v' \rangle - \langle D(x)v | B_- D(x')v' \rangle \\ &= [K_1(x, x')](v, v') - [K_1(x, x')](v, v') = 0 \end{aligned}$$

for all $(x, x') \in Z$ and $v, v' \in V$. Since $B = 0$ if and only if $B_+ = B_-$ if and only if $K_+ = K_-$ it follows that one must have $B \neq 0$.

Suppose then that there exists a nonzero $B \in L(M)$, $B^* = B$, which satisfies the condition of the theorem. We may assume that $\|B\| \leq 1$ (otherwise redefine B to be $\|B\|^{-1}B$). Since $\pm B \leq \|B\|I_M$ it follows that

$$B_{\pm} := I_M \pm B \geq 0, \quad B_+ \neq B_-.$$

Define $K_{\pm} \in \mathbf{K}(X, V)$ by

$$[K_{\pm}(x, x')](v, v') := \langle D(x)v | B_{\pm} D(x')v' \rangle, \quad x, x' \in X, v, v' \in V,$$

for which $K_+ \neq K_-$, $K = \frac{1}{2}K_+ + \frac{1}{2}K_-$, and

$$[K_{\pm}(x, x')](v, v') = \langle D(x)v | D(x')v' \rangle \pm \underbrace{\langle D(x)v | BD(x')v' \rangle}_{=0} = [K_1(x, x')](v, v')$$

for all $(x, x') \in Z$ and $v, v' \in V$. Hence, $K_{\pm} \in \mathbf{C}(K_1, Z)$ and K is not extreme. \square

Note that, if the set $Z \subseteq X \times X$ of the preceding theorem is symmetric (i.e. $(x, x') \in Z$ implies $(x', x) \in Z$) then the condition applied for *any* $B \in L(M)$ implies $B = 0$ if and only if K is extreme (if $B^* \neq B$ redefine B to be $i(B - B^*)$ which also satisfies the condition since Z is symmetric).

Example 11 (Autocorrelation functions). Let $A = \mathbb{C}$ and $K \in \mathbf{K}(X, V)$ with a MKD (M, D) so that

- V is a vector space (i.e. a \mathbb{C} -module),
- $S_{\mathbb{C}}(V)$ consists of sesquilinear forms $V \times V \rightarrow \mathbb{C}$,
- K is regular (since $\dim \mathbb{C} = 1 < \infty$),

- M is a Hilbert space (i.e. a Hilbert C^* -module over \mathbb{C}),
- any $D(x) : V \rightarrow M$ is linear.

Suppose further that $V = \mathbb{C}$ so that

$$[K(x, x')](c, c') = \langle D(x)c | D(x')c' \rangle = \bar{c} \langle D(x)1 | D(x')1 \rangle c', \quad x, x' \in X, \quad c, c' \in \mathbb{C},$$

and K can be identified with the positive definite \mathbb{C} -kernel

$$k : X \times X \rightarrow \mathbb{C}, \quad (x, x') \mapsto k(x, x') := [K(x, x')](1, 1) = \langle m(x) | m(x') \rangle$$

where $m : X \rightarrow M$, $x \mapsto m(x) := D(x)1$. We say that (M, m) is a MKD of k (i.e. the set of the linear combinations of vectors $m(x)$ is dense in the Hilbert space M). Let $K_1 \in \mathbf{K}(X, \mathbb{C})$ be defined by a MKD (M_1, D_1) where $M_1 = \mathbb{C}$ and $D_1(x) : \mathbb{C} \rightarrow \mathbb{C}$, $c \mapsto c$, for all $x \in X$. Then the corresponding positive semidefinite function is the constant function $k_1(x, x') \equiv 1$. If $Z = \{(x, x) \in X \times X \mid x \in X\}$ then $\mathbf{C}(K_1, Z)$ can be identified with the convex set $\mathbf{C}_1(X)$ of positive semidefinite functions $k : X \times X \rightarrow \mathbb{C}$ with the unit diagonal (i.e. $k(x, x) \equiv 1$). In the context of stochastic processes, such a k is called an *autocorrelation function*, and the characterization of extreme autocorrelation functions has long been a problem (see, e.g. [29, 27] and references therein). The next immediate corollary of Theorem 7 solves this problem completely.

Proposition 4. *Let $k \in \mathbf{C}_1(X)$ and (M, m) its MKD. Then k is extreme in $\mathbf{C}_1(X)$ if and only if, for any bounded operator $B : M \rightarrow M$, the condition*

$$\langle m(x) | Bm(x) \rangle = 0, \quad x \in X,$$

implies $B = 0$.

The above proposition is a generalization of Theorem 1 of [27] and hence a generalization of Theorem 1 of [29]. Following [29, 27] we see that there exist an extreme autocorrelation function m of any rank $r := \dim M \leq \sqrt{\#X}$.

7.2. Completely positive maps. Let $E : A_e \rightarrow S_A(V)$ be a \mathbb{C} -linear mapping. For any $n \in \mathbb{N}_+$ we define the n^{th} amplification $E^{(n)} : M_n(A_e) \rightarrow M_n(S_A(V))$ of E as $E^{(n)}((b_{ij})_{i,j}) := (E(b_{ij}))_{i,j}$. For example, $E^{(1)} = E$. We say that $E^{(n)}$ is positive if $E^{(n)}((b_{ij})_{i,j})$ is positive for any positive $(b_{ij})_{i,j}$. Moreover, E is *completely positive (CP)* if $E^{(n)}$ is positive for all $n \in \mathbb{N}_+$. It is easy to see that E is CP if and only if $\tilde{E} : A_e \times A_e \rightarrow S_A(V)$, $(b, b') \mapsto E(b^*b')$ is a positive-definite A -kernel, i.e.

$$\sum_{i,j=1}^n E(b_i^*b_j)(v_i, v_j) \geq 0$$

for all $n \in \mathbb{N}_+$, $b_1, \dots, b_n \in A_e$, and $v_1, \dots, v_n \in V$.

We have proved the following generalization of Kasparov-Stinespring-Gelfand-Naimark-Segal (KSGNS) theorem [34]:

Theorem 8. *Let $E : A_e \rightarrow S_A(V)$ be a CP (\mathbb{C} -linear) map. There exist a Hilbert C^* -module M over A , a unital $*$ -homomorphism $\pi : A_e \rightarrow L(M)$, and an A -linear $Y : V \rightarrow M$ such that*

- (i) $E(b)(v, v') = \langle Yv | \pi(b)Yv' \rangle$, $b \in A_e$, $v, v' \in V$,
- (ii) $\text{lin}_{\mathbb{C}} \pi(A_e)YV = \text{lin}_{\mathbb{C}} \{\pi(b)Yv \mid b \in A_e, v \in V\}$ is dense in M .

We say that (M, π, Y) is a minimal dilation for E .

If (M', π', Y') is another minimal dilation for E then there is a unitary mapping $U : M \rightarrow M'$ such that

$$\pi'(b) = U\pi(b)U^*, \quad b \in A_e,$$

and $Y' = UY$.

Note that a MKD (M, D) of \tilde{E} is related with a minimal dilation (M, π, Y) of E . Indeed, as shown in [34], one may choose $Y = D(e)$ and $\pi(b)Y = D(b)$ so that $[\tilde{E}(b, b')](v, v') = \langle D(b)v | D(b')v' \rangle = \langle \pi(b)Yv | \pi(b')Yv' \rangle = \langle Yv | \pi(b)^* \pi(b')Yv' \rangle = \langle Yv | \pi(b^*b')Yv' \rangle = E(b^*b')(v, v')$. We say that E is regular if \tilde{E} is regular.

Denote by $\mathbf{CP}(A_e, V)$ the convex set of CP maps $E : A_e \rightarrow S_A(V)$ and fix an $E_1 \in \mathbf{CP}(A_e, V)$ and $Z = \{(b, b') \in A_e \times A_e \mid b^*b' = e\}$. Denote $s_e := E_1(e) \in S_A(V)$ and define a convex set

$$\mathbf{CP}(A_e, V, s_e) := \{E \in \mathbf{CP}(A_e, V) \mid E(e) = s_e\} \subseteq \mathbf{C}(\tilde{E}_1, Z)$$

From Theorem 7 one gets that a regular $E \in \mathbf{CP}(A_e, V, s_e)$ is extreme in $\mathbf{C}(\tilde{E}_1, Z)$ if and only if, for all $B \in L(M)$, the condition

$$\langle \pi(b)Yv | B\pi(b')Yv' \rangle = 0, \quad b, b' \in A_e, \quad b^*b' = e, \quad v, v' \in V,$$

implies $B = 0$. The next theorem characterizes completely the regular extreme points of the smaller set $\mathbf{CP}(A_e, V, s_e)$ and is thus a generalization of [1, Theorem 1.4.6]:

Theorem 9. *Let $E \in \mathbf{CP}(A_e, V, s_e)$ be regular and (M, π, Y) its minimal dilation. Then $K \in \text{Ext } \mathbf{CP}(A_e, V, s_e)$ if and only if, for all $B \in L(M)$ commuting with π (i.e. $[B, \pi(b)] = 0$ for all $b \in A_e$) the condition*

$$\langle Yv | BYv' \rangle = 0, \quad v, v' \in V,$$

implies $B = 0$.

Proof. We follow the proof of Theorem 7 and use the same notations defined there: Assume that $E = \frac{1}{2}E_+ + \frac{1}{2}E_-$, $E_\pm \in \mathbf{CP}(A_e, V, s_e)$, $E_+ \neq E_-$, and denote $K = \tilde{E}$ and $K_\pm = \tilde{E}_\pm$ (so that $\tilde{E} = \frac{1}{2}\tilde{E}_+ + \frac{1}{2}\tilde{E}_-$). Now

$$E_\pm(b^*b')(v, v') = [\tilde{E}_\pm(b, b')](v, v') = \langle \pi(b)Yv | B_\pm \pi(b')Yv' \rangle, \quad b, b' \in B, \quad v, v' \in V,$$

where $B_\pm \in L(M)$ are positive, and one gets

$$E_\pm(b^*b''b')(v, v') = [\tilde{E}_\pm((b'')^*b, b')](v, v') = [\tilde{E}_\pm(b, b''b')](v, v'),$$

that is,

$$\langle \pi(b)Yv | (B_\pm \pi(b'') - \pi(b'')B_\pm) \pi(b')Yv' \rangle = 0, \quad b, b', b'' \in B, \quad v, v' \in V.$$

By density, $[B_\pm, \pi(b'')] = 0$ for all $b'' \in B$ and $B := B_+ - B_- \neq 0$ commutes with π . Moreover, $\langle Yv | BYv' \rangle = E_+(e)(v, v') - E_-(e)(v, v') = s_e(v, v') - s_e(v, v') = 0$.

Suppose then that there exists a nonzero $B \in L(M)$ which satisfies the condition of the theorem. We may assume that $B^* = B$ and $\|B\| \leq 1$ so that $B_\pm := I_M \pm B \geq 0$, $B_+ \neq B_-$, and B_\pm commute with π . Define $E_\pm \in \mathbf{CP}(A_e, V, s_e)$ by

$$E_\pm(b)(v, v') := \langle Yv | B_\pm \pi(b)Yv' \rangle, \quad b \in B, \quad v, v' \in V,$$

for which $E_+ \neq E_-$ and $E = \frac{1}{2}E_+ + \frac{1}{2}E_-$ so that E is not extreme. \square

Example 12 (Instruments). Let $\mathcal{M} : \Sigma \times \mathcal{L}(\mathcal{K}) \rightarrow \mathcal{L}(\mathcal{H})$ be a CP instrument and $\mu : \Sigma \rightarrow [0, \infty]$ a σ -finite measure such that \mathcal{M} is absolutely continuous with respect to it. Denote, by the same symbol \mathcal{M} , the CP ‘extension’ (or linearization) of \mathcal{M} to the tensor product (von Neumann) algebra $\mathcal{L}(\mathcal{K}) \otimes L^\infty(\mu)$ of von Neumann algebras $\mathcal{L}(\mathcal{K})$ and $L^\infty(\mu)$ (with the preduals $\mathcal{T}(\mathcal{K})$ and $L^1(\mu)$, respectively). Recall that $\mathcal{L}(\mathcal{K}) \otimes L^\infty(\mu)$ can be viewed as the (equivalence classes of) μ -measurable field of operators, $\Omega \ni x \mapsto B(x) \in \mathcal{L}(\mathcal{K})$, such that $\|B\| := \mu\text{-ess sup}_{x \in \Omega} \|B(x)\| < \infty$.

In Theorems 8 ja 9, $E = \mathcal{M}$, $A_e = \mathcal{L}(\mathcal{K}) \otimes L^\infty(\mu)$, $A = \mathbb{C}$, $V = \mathcal{H}$, $s_e = I_{\mathcal{H}}$, M is a Hilbert space, $Y : \mathcal{H} \rightarrow M$ is a linear isometry, and $\pi : \mathcal{L}(\mathcal{K}) \otimes L^\infty(\mu) \rightarrow \mathcal{L}(M)$ a unital $*$ -homomorphism. As shown in the proof of Theorem 2, M can be chosen to be $M = \mathcal{K} \otimes \mathcal{H}_\oplus$ and $\pi(B \otimes f) = B \otimes \hat{f}$.

For Theorem 2 one gets an interesting observation: Let

$$\langle \varphi | \mathcal{M}(X, B) \psi \rangle = \int_X \sum_{k=1}^{n(x)} \langle \varphi | \mathbf{A}_k(x)^* B \mathbf{A}_k(x) \psi \rangle d\mu(x), \quad \varphi, \psi \in V_{\mathbf{h}},$$

be a minimal pointwise Kraus form of \mathcal{M} . For all $x \in \Omega$, define a rank $n(x)$ CP map

$$\mathcal{M}(x, \bullet) : \mathcal{L}(\mathcal{K}) \rightarrow S_{\mathbb{C}}(V_{\mathbf{h}}), \quad B \mapsto \mathcal{M}(x, B),$$

where

$$\mathcal{M}(x, B)(\varphi, \psi) := \sum_{k=1}^{n(x)} \langle \varphi | \mathbf{A}_k(x)^* B \mathbf{A}_k(x) \psi \rangle, \quad \varphi, \psi \in V_{\mathbf{h}},$$

so that $\langle \varphi | \mathcal{M}(X, B) \psi \rangle = \int_X \mathcal{M}(x, B)(\varphi, \psi) d\mu(x)$ and we have obtained a CP density of a CP instrument. Conversely, if a σ -finite positive measure μ and an ON basis $\mathbf{h} \subset \mathcal{H}$ are given, then any (weakly μ -measurable) family of CP maps $E_x : \mathcal{L}(\mathcal{K}) \rightarrow S_{\mathbb{C}}(V_{\mathbf{h}})$ defines an instrument if and only if $\int_{\Omega} E_x(I_{\mathcal{K}})(h_n, h_m) d\mu(x) = \delta_{nm}$.

Remark 7 (Choi isomorphism). In this remark we generalize Choi's Theorem 2 of [8] which forms a basis for Choi isomorphism widely used in quantum information theory.

Let $\mathbf{k} = \{k_s\}$ be a ON basis of \mathcal{K} , V a module over a C^* -algebra A , and $E : \mathcal{L}(\mathcal{K}) \rightarrow S_A(V)$ a \mathbb{C} -linear map. Let $X := \{s \in \mathbb{N}_+ \mid s < \dim \mathcal{K} + 1\}$ and define a mapping $K_E : X \times X \rightarrow S_A(V)$ by $K_E(s, t) := E(|k_s\rangle\langle k_t|)$. We say that K_E is a *matrix of E* (with respect to the basis \mathbf{k}) and denote briefly $K_E \equiv \left(E(|k_s\rangle\langle k_t|) \right)_{s,t=1}^{\dim \mathcal{K}}$. Moreover, if K_E is a positive-definite A -kernel, that is, for all $n < \dim \mathcal{K} + 1$ and $v_1, \dots, v_n \in V$,

$$(5) \quad \sum_{s,t=1}^n [E(|k_s\rangle\langle k_t|)](v_s, v_t) \geq 0,$$

we say that K_E is *positive-(semi)definite matrix*. Equip $\mathcal{L}(\mathcal{K})$ with the (ultra)weak topology and $S_A(V)$ with the locally convex topology generated by the seminorms $s \mapsto \|s(v_1, v_2)\|$, $v_1, v_2 \in V$. Now we have the following generalization of [8, Theorem 2]:

Theorem 10. *Let $E : \mathcal{L}(\mathcal{K}) \rightarrow S_A(V)$ be a continuous \mathbb{C} -linear map. Then $E \in \mathbf{CP}(\mathcal{L}(\mathcal{K}), V)$ if and only if a matrix K_E of E is positive definite.*

Proof. Now $E \in \mathbf{CP}(\mathcal{L}(\mathcal{K}), V)$ if and only if

$$(6) \quad \sum_{i,j=1}^n E(b_i^* b_j)(v_i, v_j) \geq 0$$

for all $n \in \mathbb{N}_+$, $b_1, \dots, b_n \in \mathcal{L}(\mathcal{K})$, and $v_1, \dots, v_n \in V$. Obviously, (5) follows from (6) by setting $b_j \equiv |k_1\rangle\langle k_j|$. Suppose then that (5) holds. Then (6) clearly holds for operators b_i whose matrices are finite (i.e. $\langle k_s | b_i k_t \rangle \neq 0$ for finitely many indices s and t). The $*$ -subalgebra $\mathcal{L}(\mathcal{K})_{\text{fin}}$ consisting of operators with finite matrices is (ultra)weakly dense in $\mathcal{L}(\mathcal{K})$ so that, by continuity of E , the condition (6) must hold for all operators $b_i \in \mathcal{L}(\mathcal{K})$. \square

REFERENCES

- [1] W. B. Arveson, "Subalgebras of C^* -algebras," Acta Math. **123**, 141-224 (1969).
- [2] P. Busch and P. J. Lahti, "Completely positive mappings in quantum dynamics and measurement theory," Found. Phys. **20**, 1429-1439 (1990).
- [3] P. Busch and P. J. Lahti, "The standard model of quantum measurement theory: history and applications," Found. Phys. **26**, 875-893 (1996).

- [4] P. Busch, P. J. Lahti, and P. Mittelstaedt, *The Quantum Theory of Measurement – Second Revised Edition* (Springer-Verlag, Berlin, 1996).
- [5] C. Carmeli, T. Heinosaari, and A. Toigo, "Sequential measurements of conjugate observables," *J. Phys. A: Math. Theor.* **44** 285304 (2011).
- [6] U. Cattaneo, "On Mackey's imprimitivity theorem," *Comment. Math. Helvetici* **54**, 629-641 (1979).
- [7] G. Chiribella, G. M. D'Ariano, and P. Perinotti, "Realization schemes for quantum instruments in finite dimensions," *J. Math. Phys.* **50**, 042101 (2009).
- [8] M. Choi, "Completely positive linear maps on complex matrices," *Linear Algebra Appl.* **10**, 285-290 (1975).
- [9] G. M. D'Ariano, P. Perinotti, and M. Sedlák, "Extremal quantum protocols," *J. Math. Phys.* **52**, 082202 (2011).
- [10] E. B. Davies, *Quantum Theory of Open Systems* (Academic Press, London, 1976).
- [11] E. B. Davies and J. T. Lewis, "An operational approach to quantum probability," *Commun. Math. Phys.* **17**, 239-260 (1970).
- [12] J. Dixmier, *Von Neumann Algebras* (North-Holland, Amsterdam, 1981).
- [13] J. M. Gracia-Bondia, J. C. Varilly, and H. Figueroa, *Elements of Noncommutative Geometry* (Birkhäuser Advanced Texts: Basler Lehrbücher, Birkhäuser Boston, Inc., Boston, 2001).
- [14] E. Haapasalo, T. Heinosaari, and J.-P. Pellonpää, "Quantum measurements on finite dimensional systems: relabeling and mixing," *Quantum Information Processing*, in press, arXiv:1104.4886.
- [15] E. Haapasalo, P. Lahti, and J. Schultz, "Weak versus approximate values in quantum state determination," *Phys. Rev. A* **84**, 052107 (2011).
- [16] E. Hawkins, F. Markopoulou, and H. Sahlmann, "Evolution in quantum causal histories," *Class. Quant. Grav.* **20**, 3839-3854 (2003).
- [17] T. Heinosaari and M. M. Wolf, "Non-disturbing quantum measurements," *J. Math. Phys.* **51**, 092201 (2010).
- [18] T. Heinosaari and M. Ziman, *The Mathematical Language of Quantum Theory – From Uncertainty to Entanglement* (Cambridge University Press, Cambridge, 2012).
- [19] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (North-Holland, Amsterdam, 1982); *Statistical structure of quantum theory* (Lecture Notes in Physics 67, Springer-Verlag, Berlin, 2001).
- [20] A. S. Holevo, "Radon–Nikodym derivatives of quantum instruments," *J. Math. Phys.* **39**, 1373-1387 (1998).
- [21] A. S. Holevo, "Entanglement-breaking channels in infinite dimensions," *Problems of Information Transmission* **44**, 171-184 (2008).
- [22] A. S. Holevo, "On extreme quantum Gaussian channels," arXiv:1111.3552.
- [23] A. S. Holevo, M. E. Shirokov, and R. F. Werner, "On the notion of entanglement in Hilbert spaces," *Russian Math. Surveys* **60**, 359-360 (2005).
- [24] M. Horodecki, P. W. Shor, and M. B. Ruskai, "Entanglement breaking channels," *Rev. Math. Phys.* **15**, 629-641 (2003).
- [25] T. Hytönen, J.-P. Pellonpää, and K. Ylinen, "Positive sesquilinear form measures and generalized eigenvalue expansions," *J. Math. Anal. Appl.* **336**, 1287-1304 (2007).
- [26] T. Hytönen, J.-P. Pellonpää, and K. Ylinen, "Diagonalization and representation results for nonpositive sesquilinear form measures," *J. Math. Anal. Appl.* **338**, 716-725 (2008).
- [27] J. Kiukas and J.-P. Pellonpää, "A note on infinite extreme correlation matrices," *Linear Algebra Appl.* **428**, 2501-2508 (2008).
- [28] K. Kraus, *States, Effects, and Operations* (Springer-Verlag, Berlin, 1983).
- [29] C.-K. Li and B.-S. Tam, "A note on extreme correlation matrices," *SIAM J. Matrix Anal. Appl.* **15**, 903-908 (1994).
- [30] V. M. Manuilov and E. V. Troitsky, *Hilbert C^* -modules*. Translated from the 2001 Russian original by the authors. (Translations of Mathematical Monographs, 226. American Mathematical Society, Providence, RI, 2005).
- [31] M. Ozawa, "Quantum measuring processes of continuous observables," *J. Math. Phys.* **25**, 79-87 (1984).
- [32] M. Ozawa, "Concepts of conditional expectations in quantum theory," *J. Math. Phys.* **26**, 1948-1955 (1985).
- [33] J.-P. Pellonpää, "Complete characterization of extreme quantum observables in infinite dimensions," *J. Phys. A: Math. Theor.* **44**, 085304 (2011).
- [34] J.-P. Pellonpää and K. Ylinen, "Modules, completely positive maps, and a generalized KSGNS construction," *Positivity* **15**, 509-525 (2011).
- [35] W. F. Stinespring, "Positive functions on C^* -algebras," *Proc. Amer. Math. Soc.* **6**, 211-216 (1955).
- [36] S.-K. Tsui, "Complete positive module maps and completely positive extreme maps," *Proc. AMS* **124**, 437-445 (1996).

[37] K. Ylisen, "Quantum instruments and related transformation valued functions," Found. Phys. **39**, 656-675 (2009).

APPENDIX

Remark 8. The vectors and operators of Theorem 2 have the following interrelations: Let $\mathcal{M} \in \text{Ins}(\Sigma, \mathcal{K}, \mathcal{H})$ be an instrument with the *structure vectors* $\psi_n^s \in \mathcal{H}_\oplus \subseteq L^2(\mu, \mathcal{L})$ associated with the ON bases $\{h_n\} \subset \mathcal{H}$ and $\{k_s\} \subset \mathcal{K}$. Let $\{b_k\}$ be an ON basis of \mathcal{L} . Then⁵

$$\begin{aligned}
\langle b_k | \psi_n^s(x) \rangle &= \langle d_k^s(x) | h_n \rangle = \langle k_s | A_k(x) h_n \rangle = \langle b_k | A^s(x) h_n \rangle, \\
\psi_n^s(x) &= \sum_k \langle d_k^s(x) | h_n \rangle b_k = \sum_k \langle k_s | A_k(x) h_n \rangle b_k = A^s(x) h_n, \\
d_k^s(x) &= A_k(x)^* k_s = A^s(x)^* b_k = \sum_n \langle \psi_n^s(x) | b_k \rangle h_n, \\
A_k(x) &= \sum_s |k_s\rangle \langle b_k | A^s(x) = \sum_{s,n} \langle b_k | \psi_n^s(x) \rangle |k_s\rangle \langle h_n| = \sum_s |k_s\rangle \langle d_k^s(x)|, \\
A^s(x) &= \sum_n |\psi_n^s(x)\rangle \langle h_n| = \sum_k |b_k\rangle \langle d_k^s(x)| = \sum_k |b_k\rangle \langle k_s | A_k(x), \\
(Yh_n)(x) &= \sum_s k_s \otimes \psi_n^s(x) = \sum_{s,k} \langle d_k^s(x) | h_n \rangle k_s \otimes b_k \\
&= \sum_k (A_k(x) h_n) \otimes b_k = \sum_s k_s \otimes (A^s(x) h_n), \\
\mathcal{M}(\Omega, I_{\mathcal{K}}) &= I_{\mathcal{H}} = Y^* Y \supseteq \sum_{s=1}^{\dim \mathcal{K}} \int_{\Omega} A^s(x)^* A^s(x) d\mu(x) = \int_{\Omega} \sum_{k=1}^{n(x)} A_k(x)^* A_k(x) d\mu(x).
\end{aligned}$$

Note that, since $\mathcal{H}_\oplus \subseteq L^2(\mu, \mathcal{L})$, $n(x) \leq \dim \mathcal{H} \dim \mathcal{K}$, and \mathcal{L} is arbitrary, it is sometimes notationally convenient to replace \mathcal{L} [resp. $\{b_k\}_{k=1}^\infty$] with $\mathcal{H} \otimes \mathcal{K}$ [resp. $\{h_n \otimes k_s\}$]. For example, when $\mathcal{M} \in \text{Ins}(\Sigma, \mathcal{K}, \mathcal{H})$ is concentrated on a point $y \in \Omega$, that is, the associated measure $\mu = \delta_y$ (the Dirac measure), then

$$\mathcal{M}(X, B) = \sum_{n,m} \sum_{s,t} B_{st} \langle \psi_n^s | \psi_m^t \rangle |h_n\rangle \langle h_m|, \quad y \in X,$$

where $\psi_n^s \in \mathcal{H} \otimes \mathcal{K}$. By choosing $\psi_n^s = \sqrt{\lambda_s} h_n \otimes k_s$, $\lambda_s \geq 0$, $\sum_s \lambda_s = 1$ one gets a channel $\mathcal{M}(\Omega, B) = (\sum_s \lambda_s B_{ss}) I_{\mathcal{H}} = \text{tr}[\lambda B] I_{\mathcal{H}}$ where $\lambda = \sum_s \lambda_s |k_s\rangle \langle k_s|$.

Remark 9. Let $\mathcal{M}(X, B) = Y^*(B \otimes \hat{\chi}_X)Y = \tilde{Y}^*(B \otimes \hat{\chi}_X)\tilde{Y}$ be two minimal Stinespring dilations for an instrument \mathcal{M} . Here $Y : \mathcal{H} \rightarrow \mathcal{K} \otimes \mathcal{H}_\oplus$ and $\tilde{Y} : \mathcal{H} \rightarrow \mathcal{K} \otimes \widetilde{\mathcal{H}_\oplus}$ are isometries where $\mathcal{H}_\oplus = \int_{\Omega}^{\oplus} \mathcal{H}_{n(x)} d\mu(x)$ and $\widetilde{\mathcal{H}_\oplus} = \int_{\Omega}^{\oplus} \mathcal{H}_{\tilde{n}(x)} d\mu(x)$ are direct integral Hilbert spaces. Fix ON bases \mathbf{h} and \mathbf{k} , and write $Y = \sum_{m=1}^{\dim \mathcal{H}} \sum_{t=1}^{\dim \mathcal{K}} |k_t \otimes \psi_m^t\rangle \langle h_m|$ and $\tilde{Y} = \sum_{m=1}^{\dim \mathcal{H}} \sum_{t=1}^{\dim \mathcal{K}} |k_t \otimes \tilde{\psi}_m^t\rangle \langle h_m|$. From Theorem 8 (see Example 12) it follows that there is a unitary mapping $U : \mathcal{K} \otimes \mathcal{H}_\oplus \rightarrow \mathcal{K} \otimes \widetilde{\mathcal{H}_\oplus}$ such that, on $\mathcal{K} \otimes \widetilde{\mathcal{H}_\oplus}$,

$$B \otimes \hat{\chi}_X = U(B \otimes \hat{\chi}_X)U^*, \quad B \in \mathcal{L}(\mathcal{K}), X \in \Sigma,$$

and $\tilde{Y} = UY$. Hence, $U = I_{\mathcal{K}} \otimes V$ where $V : \mathcal{H}_\oplus \rightarrow \widetilde{\mathcal{H}_\oplus}$ is a unitary map and $\mathcal{H}_\oplus \cong \widetilde{\mathcal{H}_\oplus}$. Moreover, V commutes with the canonical spectral measure(s) and is thus decomposable:

⁵If $S_i : V_i \times V_i \rightarrow \mathbb{C}$, $i = 1, 2$, are sesquilinear forms (e.g. bounded operators) and $V_2 \subseteq V_1$ (a vector subspace), we denote $S_1 \supseteq S_2$ if $S_1(v, w) = S_2(v, w)$ for all $v, w \in V_2$.

$V = \int_{\Omega}^{\oplus} V(x) d\mu(x)$ where $V(x) : \mathcal{H}_{n(x)} \rightarrow \mathcal{H}_{\tilde{n}(x)}$ is unitary for almost all $x \in \Omega$. Especially, $\tilde{n}(x) = n(x)$ and

$$\tilde{\psi}_m^t(x) = V(x)\psi_m^t(x)$$

for almost all $x \in \Omega$. If \mathcal{H}_{\oplus} and $\widetilde{\mathcal{H}}_{\oplus}$ are embedded in $L^2(\mu, \mathcal{L})$ as before, that is, $\mathcal{H}_{\tilde{n}(x)} = \mathcal{H}_{n(x)} = \text{lin}_{\mathbb{C}}\{b_n \mid 1 \leq n \leq n(x)\}$ one can view $V(x)$ as a change of the ON basis $\{b_n\}_{n=1}^{n(x)}$ of the fiber $\mathcal{H}_{n(x)}$ at $x \in \Omega$: Let $d_k^s(x)$, $A_k(x)$, and $A^s(x)$ [resp. $\tilde{d}_k^s(x)$, $\tilde{A}_k(x)$, and $\tilde{A}^s(x)$] be the objects of Theorem 2 associated to the vectors ψ_m^t [resp. $\tilde{\psi}_m^t$]. Then

$$\begin{aligned} d_k^s(x) &:= \sum_n \langle \psi_n^s(x) | b_k \rangle h_n, & \tilde{d}_k^s(x) &:= \sum_n \langle \tilde{\psi}_n^s(x) | b_k \rangle h_n = \sum_n \langle \psi_n^s(x) | V(x)^* b_k \rangle h_n, \\ A^s(x) &= \sum_n |\psi_n^s(x)\rangle \langle h_n|, & \tilde{A}^s(x) &= \sum_n |\tilde{\psi}_n^s(x)\rangle \langle h_n| = V(x) A^s(x), \\ A_k(x) &= \sum_s |k_s\rangle \langle b_k | A^s(x), & \tilde{A}_l(x) &= \sum_s |k_s\rangle \langle b_l | \tilde{A}^s(x) = \sum_s |k_s\rangle \langle b_l | V(x) A^s(x) \\ & & &= \sum_k \langle b_l | V(x) b_k \rangle \sum_s |k_s\rangle \langle b_k | A^s(x) = \sum_k \langle b_l | V(x) b_k \rangle A_k(x) \end{aligned}$$

where $(\langle b_l | V(x) b_k \rangle)_{l,k=1}^{n(x)}$ is a unitary matrix (compare to Remark 4 of [8]). Now, for example,

$$\mathcal{M}(X, B) \supseteq \int_X \sum_{k=1}^{n(x)} A_k(x)^* B A_k(x) d\mu(x) = \int_X \sum_{l=1}^{n(x)} \tilde{A}_l(x)^* B \tilde{A}_l(x) d\mu(x).$$

Note that if the bases \mathbf{h} and \mathbf{k} are changed to ON bases \mathbf{h}' and \mathbf{k}' of \mathcal{H} and \mathcal{K} , respectively, then, e.g.

$$Y = \sum_{m=1}^{\dim \mathcal{H}} \sum_{t=1}^{\dim \mathcal{K}} |k_t \otimes \psi_m^t\rangle \langle h_m| = \sum_{n=1}^{\dim \mathcal{H}} \sum_{s=1}^{\dim \mathcal{K}} |k'_s \otimes \psi_n'^s\rangle \langle h'_n|$$

where

$$\psi_n'^s = \sum_{m=1}^{\dim \mathcal{H}} \sum_{t=1}^{\dim \mathcal{K}} \langle k'_s | k_t \rangle \psi_m^t \langle h_m | h'_n \rangle.$$

Now one can define the corresponding objects $d_k'^s(x)$, $A'^s(x)$, and $A'_k(x)$ related to bases \mathbf{h}' and \mathbf{k}' , and easily find their relations with $d_k^s(x)$, $A^s(x)$, and $A_k(x)$. If the measure μ is replaced by an equivalent measure $\bar{\mu}$, $d\bar{\mu}(x) = w(x)d\mu(x)$, then the density (the Radon-Nikodým derivative) w can be absorbed into fibers $\mathcal{H}_{n(x)}$ and the corresponding direct integral Hilbert spaces are unitarily equivalent. Thus, we have seen that Theorem 2 is essentially independent of the choices of μ , the ON bases, and the corresponding operators.

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